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The Semantics of the Predicate Calculus

1. The Rudiments of Set Theory

In the propositional calculus interpretations were defined to be assignments of truth values to sentential letters. We then defined a valid formula (a *tautology*) to be a formula true relative to every interpretation, and showed that a formula of the propositional calculus is a tautology if, and only if, it expresses a formally necessary statement form. We want to do something similar for the predicate calculus. However, because of the greater expressive power of the predicate calculus, interpretations must do more than assign truth values to sentential letters. They must also interpret relation symbols, individual constants, and quantifiers. This can be done set-theoretically. For example, quantifiers will be interpreted by giving a set of objects which constitutes the universe of discourse. To make this precise, the student must have some acquaintance with set theory. This section will discuss the rudiments of set theory, and then in section two we will return to the task of defining the concept of an interpretation for the predicate calculus.

First, what is a set? This is a question that we can only answer by giving synonyms. The terms “set”, “class”, “collection”, “group”, and “ensemble” can all be used interchangeably. A set may be defined to be any collection of objects. For example, we can talk about the set of all integers between one and seventeen. This set consists of the integers from two through sixteen, inclusive. Or we may talk about the set of all desks in a room, or the set of all buildings on the campus, or the set of all past Presidents of the United States, and so forth.

A set is a set of objects. The objects in a set are called the *members* or *elements* of the set. Given any set, A and any object x, we abbreviate “x is a member of A” by writing $x \in A$. Thus, for example, if A is the set of all past Presidents of the United States, and k stands for “John F. Kennedy”, we can write $k \in A$. Or, if N is the set of integers between one and sixteen, then we can write $6 \in N$.

Perhaps the most important property of sets is that they are *extensional*. This means that the identity of a set is determined entirely by what objects it has as members—if two sets have exactly the same members, then they are the same set. For example, the set of all integers between one and four is identical to the set of the two integers two and three, because these sets have the same members. Or the set of featherless bipeds is the same as the set of things that are either human beings or plucked birds. Using the symbols of the predicate calculus, we can express what it means to say that sets are extensional as follows: Given any two sets A and B,

$$[(\forall x)(x\in A \leftrightarrow x\in B) \rightarrow A = B]$$

Because the identity of a set is determined by its members, one convenient way to refer to a set having a small number of members is by listing its members. The customary way to do this is to enclose the list of members in braces, { and }. We can, for example, refer to the set of all integers between one and four by writing {2, 3}. It makes no difference in what order we list the members of the set. If the members are the same, the set is the same. So {2, 3} and {3,2} are the same set, and {1,2, 3}, {1, 3,2}, {2,1, 3}, {2, 3,1}, {3,1,2}, and {3,2,1} are the same set.

Sometimes a set will have just one member. For example, we can talk about the set of all integers between one and three, which is the set whose only member is two. This set can be referred to as {2}. A set having only one member is called the *unit set* of that object. Thus {2} is the unit set of two.

Sometimes a set will have too many members to make it practical to list all of them. A set might even have an infinite number of members, like the set of all integers. When this happens we cannot refer to the set by simply listing the members, but we can adapt the above notation to take care of this. If for some predicate A we want to talk about the set of all objects that are A (such as the set of all past presidents), we can simply write $\{x \mid x \text{ is } A\}$. This is read "the set of all objects, x, such that x is A". Using these symbols the set of all past presidents is $\{x \mid x \text{ is a past president}\}$.

There is one set that deserves special mention because of its somewhat paradoxical-seeming nature. This is the set that has no members. This is called the *empty set*, and is designated by the symbol $\varnothing$. The empty set can be introduced in many different ways. For example, $\varnothing = \{x \mid x \text{ both is and is not an integer}\}$ because there is nothing that both is and is not an integer. Notice that there can only be one empty set—this follows from the extensionality of sets. If we had two sets neither of which had any members, then they would have the same members (that is, $(\forall x)(x\in A \leftrightarrow x\in B)$ would be true), and so they would be the same set.

A set that is just the opposite of the empty set is the *universal set*. The universal set is the set containing everything. It contains all numbers, material objects, people, and so on. The universal set is denoted by U. We can define it as $U = \{x \mid x \text{ either is or is not an integer}\}$ because everything either is or is not an integer.

The order in which we list the members of a set makes no difference to the identity of the set. But sometimes we would like it to make a difference. This leads to the concept of an *ordered-set*, or a *sequence*. A sequence is a list of objects having a definite order. For example, we may talk about the sequence of integers from one to ten, listed in order from smallest to largest. This will be distinct from the sequence of integers from one to ten listed in order from largest to smallest.

Just as { and } are used to talk about sets, $\langle$ and $\rangle$ are used to talk about sequences. Thus, the sequence of integers from one to ten listed in order from smallest to largest will be the sequence $\langle 1,2, 3, 4, 5, 6, 7, 8, 9,10\rangle$, and the sequence of integers from one to ten listed in the reverse order will be ⟨10, 9, 8, 7, 6, 5, 4, 3,2,1⟩. These are different sequences. Unlike sets, sequences are not extensional. Whereas the identity of a set is determined solely by its members, the identity of a sequence is determined by the combination of its members and their order.

We will frequently want to talk about sequences of fixed length. A sequence of two objects will be called an *ordered pair*. Thus ⟨1,2⟩, ⟨John Kennedy, F. Scott Fitzgerald⟩, ⟨Hiroshima, $\sqrt{2}$⟩ are all ordered pairs. Similarly, a sequence of three objects is called an *ordered triple*, and a sequence of four objects is called an *ordered quadruple*. Above a certain point we run out of names like this, so we use another kind of name. An ordered pair can also be called an *ordered two-tuple*, and ordered triple can be called an *ordered three-tuple*, and so on. This terminology will work for a sequence of any length. A sequence of 137 objects can be called an ordered 137-tuple. In general, given any number n, we can talk about sequences of n objects and call them *ordered n-tuples*.

There is a close relationship between sets and relations. The simplest relations are one-place relations, or predicates. Corresponding to each predicate is the set of all objects to which that predicate can be truly ascribed. For example, corresponding to the predicate "is a bachelor" there is the set of all bachelors; that is {x | x is a bachelor}. And corresponding to the predicate "is an integer" there is the set of all integers; that is, {x | x is an integer}. In other words, given any predicate A, there corresponds to it {x | x is A}. This set is called the *extension* of the predicate A. **An object is a member of the extension of a predicate if, and only if, that predicate can be truly ascribed to that object.**

Next consider two-place relations, such as "(1) is the brother of (2)". There is also a set corresponding to this, but now it is a set of ordered pairs. For example, corresponding to the relation "(1) is the brother of (2)" is the set of all ordered pairs for which the first element of the pair is the brother of the second element of the pair. Or corresponding to the relation "(1) is an integer less than (2), and (2) is an integer less than four", we have the set of all ordered pairs of integers such that the first integer in each pair is less than the second integer, and the second integer is less than four. This is just the set {⟨1,2⟩, ⟨1, 3⟩, ⟨2, 3⟩}. In general, given a two-place relation "(1) ... (2)" (where we fill something in for the blank), we can talk about the set of all ordered pairs which are such that the first member of each pair stands in this relation to the second member of that pair. This set can be referred to as "the set of all ordered pairs, ⟨x, y⟩, such that x ... y", and can be symbolized as {⟨x, y⟩ | x ... y}. Corresponding to the relation "(1) is the brother of (2)" is {⟨x, y⟩ | x is the brother of y}. This set of ordered pairs is called the *extension* of the relation. The extension of the relation "(1) is an integer between (2) and (3), and (2) is greater than four and (3) is less than nine" is the set {⟨6, 5, 7⟩, ⟨6, 5, 8⟩, ⟨7, 5, 8⟩, ⟨7, 6, 8⟩}. **Given an n-place relation, the elements of a particular ordered n-tuple of objects stand in this relation to one another if, and only if, the ordered n-tuple is a member of the extension of the relation.**

In the next section the concept of the extension of a predicate or relation will be used to analyze the conditions under which a statement whose form can be symbolized in the predicate calculus is true.

Exercises

A. Which of the following are the same (equal)?

1. {1,2, 3}
2. {3,2}
3. {3,1,2}
4. ⟨3,1,2⟩
5. {2,(5 – 2),1}
6. ⟨1,2, 3⟩
7. {3,2, 3}
8. {1,2,2,1, 3}
9. {x | x is an integer between 0 and 4}

B. What are the extensions of the following predicates and relations?

1. “is a positive integer smaller than 7”
2. “is (or was) president of the United States after Carter”
3. “(1) was president after Carter, but before (2)”
4. “(1), (2), and (3) are positive integers smaller than four, and (1) is the sum of (2) and (3)”

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2. Interpretations

Interpretations must contain enough information to enable us to determine the truth values of formulas relative to the interpretations. For this purpose, an interpretation must specify a universe of discourse for the quantifiers, and it must interpret the sentential letters, relation symbols, and individual constants. The universe of discourse will be called *the domain of the interpretation.* In the propositional calculus, it proved unnecessary to specify the entire meaning of a sentential letter. For the purpose of computing truth values, it was sufficient to simply assign truth values to the sentential letters. That remains true in the predicate calculus, so we will take an interpretation of the predicate calculus to assign truth values to sentential letters. Similarly, an interpretation will assign a *denotation* to each individual constant. The denotation of an individual constant is the object it denotes or names.

A one-place relation symbol is used to express a predicate. The statement symbolized by Fc is true if, and only if, the predicate expressed by F can be correctly ascribed to the object denoted by c. But as we saw in section one, that condition is satisfied if, and only if, the object denoted by c is in the extension of the predicate expressed by F. Thus for the purpose of determining the truth value of Fc, it suffices to know the extension of F and the denotation of c. We do not have to know the “meaning” of F, in the sense of knowing what predicate it expresses.

This observation can be extended to n-ary relation symbols. For example, consider the atomic formula Hbc. Suppose b denotes an object ***b*** and c denotes an object ***c***. Then Hbc is true if, and only if, $\langle \boldsymbol{b},\boldsymbol{c} \rangle$ is in the extension of the relation expressed by H. So for the purpose of determining the truth value of Hbc, it suffices to know the extension of H and the denotations of b and c.

It will turn out that something similar is true for arbitrary formulas of the propositional calculus. All we have to know to evaluate their truth values are the universe of discourse, the denotations of individual constants, the extensions of relation symbols, and the truth values of sentential letters. Accordingly, we will take an interpretation of the predicate calculus to specify just these four items:

An *interpretation of the predicate calculus* is an assignment of the following:

1. A set of individuals to constitute the universe of discourse (this is the *domain* of the interpretation); and
2. denotations in the domain for all individual constants; and
3. extensions in the domain for all relation symbols; and
4. truth values for all sentential letters.

Note that the denotation of an individual constant must be chosen from the domain of the interpretation. Similarly, the extension of a relation symbol must consist of objects from the domain. For example, the extension assigned to a two-place relation must be a set of ordered pairs of objects in the domain. This requirement should be emphasized, because it is a common mistake to overlook it. If it were not required that the denotations and extensions be chosen from the domain, formulas would not have their intended meaning. For example, the formula $[(\forall x)Fx \rightarrow Fc]$ means “If everything in the domain, including c, is F, then c is F”. So we must ensure that the denotation of c is in the domain. Note also that because all individual constants must be assigned denotations, it follows that there must be at least one thing in the domain. That is, the domain cannot be empty.

Notice that whenever we are given a universe of discourse and “meanings” for the relation symbols and individual constants in a closed formula we can construct a corresponding interpretation. The domain will be the universe of discourse, and the extensions and denotations assigned to the relation symbols and individual constants are those determined by their meaning. In constructing an interpretation, we simply abstract from the meaning those features that are necessary in order to determine truth values.

3. Truth Rules

To compute the truth value of a formula relative to an interpretation, we need truth rules, so let us turn to the task of formulating such rules. The first thing to observe is that we cannot talk about the truth or falsity of an open formula—only of closed formulas. Open formulas symbolize relations rather than statements, and only statements are true or false. For instance, we cannot ask whether the relation (1) is the brother of (2) is true. That question makes no sense. Truth rules can only be applicable to closed formulas.

The truth rules employed in the propositional calculus are still correct for the predicate calculus. However, because of the greater expressive power the predicate calculus, those rules are not sufficient for computing truth values for all formulas. The simplest cases they omit are those of atomic formulas involving relation symbols. To compute truth values for these formulas we need the following truth rules:

(1) If F is a relation symbol and c is an individual constant, Fc is true relative to an interpretation if and only if the interpretation assigns an object $\boldsymbol{c}$ as the denotation of c, a set $\boldsymbol{F}$ of objects as the extension of F, and $\boldsymbol{c} \in \boldsymbol{F}$.

(2) If F is a relation symbol and $c_1,\ldots,c_n$ are individual constants, $Fc_1\ldots c_n$ is true relative to an interpreation if and only if the intepretation assigns objects $\boldsymbol{c_1},\ldots,\boldsymbol{c_n}$ as the denotations of $c_1,\ldots,c_n$, a set $\boldsymbol{F}$ of ordered n-tuples as the extension of F, and $\langle \boldsymbol{c_1},\ldots,\boldsymbol{c_n} \rangle \in \boldsymbol{F}$.

For example, suppose we use H to express the mathematical relation x+y = z, and restrict our domain to the positive integers less than 4. So the domain is the set {1,2,3}, and the extension of H is $\{\langle 1,1,2\rangle,\langle 1,2,3\rangle,\langle 2,1,3\rangle\}$. Then if the denotation of b is 1, c is 2, and d is 3 (so Hbc expresses the statement "1+2 = 3"), Hbc is true if, and only if, $\langle 1,2,3\rangle$ is in the extension of H. This condition is satisfied, so Hbc is true relative to this interpretation.

We will retain the following rules from the propositional calculus:

(3) An atomic closed formula that is a sentential letter is true under an interpretation if, and only if, the interpretation assigns the truth value "true" to it.

(4) If A is any closed formula, $\sim A$ is true if, and only if, A is false.

(5) If A and B are any closed formulas, $(A \,\&\, B)$ is true if and only if both A and B are true.

(6) If A and B are any closed formulas, $(A \vee B)$ is true if and only if either A or B (or both) are true.

(7) If A and B are any closed formulas, $(A \rightarrow B)$ is true if and only if either A is false or B is true.

(8) If A and B are any closed formulas, $(A \leftrightarrow B)$ is true if and only A and B have the same truth value, i.e., either both are true or both are false.

Using the preceding truth rules we can determine the truth value of any closed formula that does not contain quantifiers. For example, suppose we want to know the truth value of the closed formula $[P \leftrightarrow (Qa \,\&\, \sim Fab)]$ under the interpretation that assigns the domain {1,2, 3}, the denotation 1 to a and the denotation 2 to b, the extension {2, 3} to Q, and the extension {⟨1,2⟩, ⟨2, 3⟩} to F, and the truth value "true" to P. Then by Rule 3, P is true under the interpretation. The denotation of a is 1, and this is not a member of the extension of Q, so by Rule 1, Qa is false under the interpretation. The ordered pair consisting of the denotations of a and b is ⟨1,2⟩. This ordered pair is a member of the extension of F, so by Rule 2 Fab is true under the interpretation. Then by Rule 4, ~Fab is false under the interpretation. Both Qa and ~Fab are false, so by Rule 5, (Qa & ~Fab) is false under the interpretation. P is true, so then by Rule 8 the whole closed formula [P ↔ (Qa & ~Fab)] is false under the interpretation.

We have yet to construct rules for determining whether universal generalizations and existential generalizations are true. These rules are more complicated than the above. To see how these rules must read, let us begin by looking at a simple case—the formula (∀x)Fx. This formula is true if, and only if, F is true of everything in the domain. That is, (∀x)Fx says that everything in the domain is an F. One way of making this more precise is the following. Let c be an arbitrarily chosen individual constant. Then (∀x)Fx is true if, and only if, Fc would be true regardless of what (in the domain) we let the denotation of c be. If the domain is {1,2, 3}, then (∀x)Fx is true if, and only if, each of 1,2, and 3 are in the extension of F. And the latter will be true provided that Fc would be true regardless of which of 1,2, or 3 we assign as the denotation of c.

To formulate a precise truth rule for universal generalizations, we need some way of talking about the result of varying the denotations of individual constants. Given one interpretation, let us say that another interpretation is an c-variant of the first interpretation if, and only if, the second interpretation is exactly like the first interpretation except possibly for the denotation it assigns to c. Then to talk about different c-variants of an interpretation is just a way of letting the denotation of c vary over the different elements of the domain. If we consider an interpretation that assigns the domain {1,2, 3} and assigns the extension {1, 3} to F, there will be three possible c-variants of this interpretation. These will be those interpretations that assign the same domain to the quantifiers and the same extension to F, and that assign the denotations 1,2, or 3 to c. Then to say that Fc would be true regardless of what we take the denotation of a to be means that Fc is true under each of the c-variants of the original interpretation.

We can also talk about b-variants of an interpretation, d-variants, and so on for each individual constant. These will be those interpretations that are exactly like the initial interpretation except that they also assign a denotation to the individual constant in question.

W can formulate a preliminary version of a truth rule for universal generalizations as follows:

> A universal generalization (∀x)Fx is true under an interpretation if, and only if, Fc is true under every c-variant of the initial interpretation.

To apply the analogous rule to a more complex universal generalization, e.g., (∀x)[Fx ↔ (Gx ∨ Ha)], we choose some individual constant that does not occur in the formula, e.g., c, and construct the formula $[Fc \leftrightarrow (Gc \vee Ha)]$. Then $(\forall x)[Fx \leftrightarrow (Gx \vee Ha)]$ is true under an interpretation if, and only if, $[Fc \leftrightarrow (Gc \vee Ha)]$ is true under every c-variant of the interpretation. Notice that it makes no difference which individual constant we choose, just as long as it does not already occur in the formula.

The formula $[Fc \leftrightarrow (Gc \vee Ha)]$ is obtained by *substituting* c for every occurrence of x in $[Fx \leftrightarrow (Gx \vee Ha)]$. A complication arises from the fact that a formula can contain more than one quantifier binding the same variable. For example, in computing the truth value of $(\forall x)[Fx \leftrightarrow (Gx \vee (\exists x)Hx)]$, we would construct the formula $[Fc \leftrightarrow (Gc \vee (\exists x)Hx)]$ and ask whether it is true under every c-variant. $[Fc \leftrightarrow (Gc \vee (\exists x)Hx)]$ is constructed by substituting c for every *free occurrence* of x in $[Fx \leftrightarrow (Gx \vee (\exists x)Hx)]$. The bound occurrences are left unchanged. It is convenient to introduce some notation for talking about substitution. Let $\mathrm{Sb}(c/x)P$ be the result of substituting c for every free occurrence of x in the formula P. Then we can formulate a precise truth rule for universal generalizations as follows:

(9) A universal generalization $(\forall x)P$ is true under an interpretation if, and only if, when we choose some individual constant c that does not occur in P, $\mathrm{Sb}(c/x)P$ is true under every c-variant of the initial interpretation.

That $\mathrm{Sb}(c/x)P$ is true under every c-variant of the initial interpretation is simply another way of saying that $\mathrm{Sb}(c/x)P$ would be true regardless of what we let the denotation of c be.

Some examples follow of the use of this rule to determine whether closed formulas of the predicate calculus are true under interpretations. Consider first the closed formula $(\forall x)(Fx \rightarrow Gx)$ and the interpretation in which the domain is the set of integers $\{1,2, 3\}$, the extension assigned to G is $\{1,2\}$, and the extension assigned to F is $\{2\}$. By Rule 9, $(\forall x)(Fx \rightarrow Gx)$ is true under this interpretation if, and only if, $(Fc \rightarrow Gc)$ is true under every c-variant of this interpretation. There are three things in the domain, so there are three c-variants—we can let the denotation of c be either 1, or 2, or 3. So let us consider each one separately and show that $(Fc \rightarrow Gc)$ is true under every one of them. First let the denotation of c be 1. Gc is true under that interpretation if, and only if, $1 \in \{1,2\}$. This is the case, so Gc is true. Fc is true under this c-variant if, and only if, $1 \in \{2\}$, which is not the case. So Fc is false. Then, by Rule 7 $(Fc \rightarrow Gc)$ is true under this c-variant. Next consider the c-variant that assigns the denotation 2 to c. Gc is true under that c-variant because $2 \in \{1,2\}$. So by Rule 7, $(Fc \rightarrow Gc)$ is true under the second c-variant. The third (and last) c-variant assigns 3 to c. Both Fc and Gc are false under this c-variant, because it is not the case that $3 \in \{2\}$, and it is not the case that $3 \in \{1,2\}$. Again by Rule 7, $(Fc \rightarrow Gc)$ is true. Therefore, $(Fc \rightarrow Gc)$ is true under each c-variant of our initial interpretation, and so by Rule 9, $(\forall x)(Fx \rightarrow Gx)$ is true under that interpretation.

For a more complex example, consider how we can compute the truth value of the formula

$$[(\forall x)Hx \leftrightarrow (P \rightarrow \sim(\forall x)\sim Hx)]$$

under the interpretation that assigns the domain {1,2}, and assigns "true" to P, and the extension {1} to H. First evaluate the truth value of (∀x)Hx. (∀x)Hx is true under our interpretation if, and only if, Hc is true under each c-variant of the interpretation. There are two c-variants—that assigning 1 to c, and that assigning 2 to c. Hc is true under the first c-variant because $1 \in \{1\}$, but Hc is false under the second c-variant because it is not the case that $2 \in \{1\}$. Thus Hc is not true under every c-variant, and hence (∀x)Hx is not true under the initial interpretation. Thus the left side of the biconditional is false. Next consider (∀x)~Hx. This is true under the interpretation if, and only if, ~Hc is true under each c-variant of the interpretation. But by Rule 4, ~Hc is true under an c-variant if, and only if, Hc is false under that c-variant. We have seen that Hc is true under the first c-variant and false under the second, so ~Hc is false under the first c-variant and true under the second. Therefore (∀x)~Hx is not true under the initial interpretation. P is true, so then by Rule 7, (P → ~(∀x)~Hx) is true under the interpretation. Hence we have a biconditional the left side of which is false, and the right side of which is true, so according to Rule 8 the biconditional is false under the initial interpretation.

It should be mentioned that interpretations will frequently be chosen which assign sets of integers as domains. This is only because integers are convenient to work with. These interpretations do not have any privileged status among interpretations other than their convenience. We could in principle use any set of objects for the domain of an interpretation.

Next a truth rule for existential generalizations must be constructed. The rule here is completely analogous to the case of universal generalizations. We want a closed formula of the form (∃x)Fx to be true if, and only if, there is something in the domain which is an F; that is, if, and only if, there is at least one thing in the domain such that, if we choose it as the denotation of c, Fc will be true. So our rule is the following:

(10) An existential generalization $(\exists x)P$ is true under an interpretation if, and only if, when we choose some individual constant c that does not occur in P, Sb(c/x)P is true under at least one c-variant of the interpretation.

Consider some examples of the use of Rule 10. Let us calculate the truth value of (∃x)Bxg under the interpretation that assigns the set of all past presidents of the United States as the domain, assigns George Washington as the denotation of g, and assigns to B the set of all ordered pairs of past presidents such that the first member of each ordered pair was president before the second member. Here B is the relation "(1) was president before (2)". To determine the truth value of (∃x)Bxg under this interpretation we must first choose an individual constant not occurring in Bxg. We cannot choose g, because it occurs in Bxg, so let us choose b. Then (∃x)Bxg is true under this interpretation if, and only if, Bbg is true under at least one b-variant of this interpretation. But there is no denotation we can assign to b making Bbg true, because such a denotation would have to be a president before Washington, and there was none. Therefore Bbg is not true under any b-variant of our interpretation, and so (∃x)Bxg is false under the original interpretation.

Consider a more complicated example in which one quantifier occurs within the scope of another quantifier, viz., the closed formula (∀x)(∃y)Lxy and the interpretation that assigns the domain {1,2, 3} and in which L is the "less than" relation. The extension assigned to L is {⟨1,2⟩, ⟨1, 3⟩, ⟨2, 3⟩}. (∀x)(∃y)Lxy is true under this interpretation if, and only if, (∃y)Lcy is true under every c-variant of this interpretation. There are three c-variants—those in which we assign either 1, or 2, or 3 to c. But now, (∃y)Lcy is true under a given c-variant if, and only if, Lcb is true under some b-variant of that c-variant. For each c-variant there are three b-variants of it—those assigning 1,2, or 3 to b. We can diagram the whole thing as follows:

$$\left.\begin{array}{l}
c = 1 \left\{\begin{array}{ll} b = 1 & \text{Lcb false} \\ b = 2 & \text{Lcb true} \\ b = 3 & \text{Lcb true} \end{array}\right\} (\exists y)\text{Lcy true} \\[1em]
c = 2 \left\{\begin{array}{ll} b = 1 & \text{Lcb false} \\ b = 2 & \text{Lcb false} \\ b = 3 & \text{Lcb true} \end{array}\right\} (\exists y)\text{Lcy true} \\[1em]
c = 3 \left\{\begin{array}{ll} b = 1 & \text{Lcb false} \\ b = 2 & \text{Lcb false} \\ b = 3 & \text{Lcb false} \end{array}\right\} (\exists y)\text{Lcy false}
\end{array}\right\} (\forall x)(\exists y)\text{Lxy false}$$

Lcb is true under the second and third b-variant of the first c-variant, so (∃y)Lcy is true under the first c-variant. Lcb is true under the third b-variant of the second c-variant, so (∃y)Lcy is also true under the second c-variant. But Lcb is not true under any b-variant of the third c-variant, so (∃y)Lcy is false under the third c-variant. Therefore, it is not true that (∃y)Lcy is true under every c-variant of the initial interpretation, and so (∀x)(∃y)Lxy is false under the initial interpretation.

It will often be unnecessary to work through all of the steps of the truth rules in order to determine whether a closed formula is true under a given interpretation. If we can see what a closed formula means under an interpretation we can often see immediately whether it is true. Recall the above interpretation of (∃x)Bxg. Under that interpretation, (∃x)Bxg means "There was someone who was president before Washington". Seeing this, we know immediately that the closed formula is false. Or consider the interpretation of the closed formula (∀x)(∃y)Lxy in the preceding example. Under that interpretation this closed formula means "For every integer there is another integer such that the first is smaller than the second", that is, "For every integer there is a larger integer". If the domain had been the set of all integers this would have been true, but as the domain is just {1,2, 3} it is false—there is no integer in the domain larger than 3. Explicit appeal to the truth rules is cumbersome and often unnecessary. However, we must have such rules for those cases in which the meaning of a closed formula is so complicated that we cannot tell simply by reflection whether it is true.

Exercises

Let the universe of discourse be the set {1,2, 3}, and let G mean "is greater than one", F mean "is less than three", H mean "(1) is not equal to (2)", I mean "(1) is equal to (2)", L mean "(1) is less than (2)", P mean "one is not equal to two", and let a denote 1, b denote 2, and c denote 3. This generates the interpretation that assigns the domain {1,2, 3} to the quantifiers, the extensions {1,2} to F, {2, 3} to G, {⟨1,2⟩, ⟨1, 3⟩, ⟨2,1⟩, ⟨2, 3⟩, ⟨3,1⟩, ⟨3,2⟩} to H, {⟨1,1⟩, ⟨2,2⟩, ⟨3, 3⟩} to I, and {⟨1,2⟩, ⟨1, 3⟩, ⟨2, 3⟩} to L, and assigns "true" to P,1 to a,2 to b, and 3 to c. Under this interpretation, express the meaning of each of the following closed formulas in idiomatic English and determine whether it is true:

1. Fc
2. [(Fa & Fb) & Fc]
3. (∀x)Fx
4. (∃x)(∀y)(Fx → Fy)
5. (∃x)Gx
6. (Iab ∨ ~Iab)
7. (Iab ∨ ~Hab)
8. (∀x)(∃y)Ixy
9. (∃y)(∀x)Ixy
10. ((∀x)Ixx & (∀x)~Hxx)
11. (∀x)(∀y)(Ixy → ~Hxy)
12. (∃x)(∀y)(Lxy ∨ Ixy)
13. (∀x)(∀y)(Lxy → ~Ixy)
14. [(∀x)Gx ↔ ((Ga & Gb) & Gc)]
15. (∃y)(∀x)(Lxy ∨ Icx)
16. (P ↔ (∃x)~Ixx)
17. (∀x)(∀y)(∀z)[(Lxy & Lyz) → Lxz]
18. (∀x)~Lxx
19. (∀x)(∀y)(Lxy → ~Lyx)

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4. Valid Formulas

4.1 Validity and Formal Necessity

What makes it possible to undertake a mathematically precise investigation of formal necessity in the propositional calculus is the fact that tautologicity and formal necessity coincide. That is, a formula is a tautology if, and only if, it expresses a formally necessary statement form. Because tautologicity is a mathematically precise concept, this facilitates the study of formal necessity.

We can proceed in an analogous way in the predicate calculus. A closed formula of the predicate calculus is said to be *valid* if, and only if, it is true under every interpretation. We write $\vdash A$ to indicate that a formula A is a valid. It can then be shown that a formula of the predicate calculus is valid if, and only if, it expresses a formally necessary statement form. However, the argument for this is more involved than the analogous argument for the propositional calculus.

We have seen that it is useful to be able to symbolize statements using a restricted universe of discourse. However, that creates difficulties for connecting validity and formal necessity, so let us first consider what happens when we do not use restricted universes of discourse. In other words, let us assume that the universe of discourse is the universal set. If a closed formula of the predicate calculus is true under every interpretation, then in particular it will be true under every interpretation whose domain is the universal set. Therefore, if this closed formula is what we obtain when we symbolize a statement (without using a restricted universe of discourse), then that statement must be true. The statement is made true by its logical form, so it is formally necessary. We can conclude then that if a closed formula is valid, then any statement having that form is formally necessary.

Now it must be seen whether the converse is true. Is it true that if a statement can be symbolized in the predicate calculus, and it is formally necessary, then it is valid? At first this may seem dubious, because valid formulas must be true under *all* interpretations, including those with small domains like {1,2,3}. We cannot get such a domain by symbolizing a statement while using the universal set as the universe of discourse. It is apparent that if a formula A results from symbolizing the form of a formally necessary statement P, then the formula A must be true under every interpretation whose domain is the universal set. This is because every such interpretation must correspond to a statement having the same logical form as P, and since P is formally necessary, every statement having that logical form must be true. But this only shows that A must be true under every interpretation having this one fixed domain. In order to show that A is valid we must show that it is true under every interpretation having any nonempty domain. Surprisingly, this can be shown. This turns upon what is known as the Löwenheim Theorem, which says:

> If a formula A of the predicate calculus is true under every interpretation having some particular infinite domain, then A is true under every interpretation having any nonempty domain (that is, A is valid).

The universal set is clearly infinite, because among other things it contains all numbers. Therefore, by the Löwenheim Theorem, if A is true under every interpretation whose domain is the universal set, then A is valid. And therefore, if a formula A results from symbolizing the form of a formally necessary statement, then A is valid.

Combining the above two results, we have the following: **a formula of the predicate calculus is valid if, and only if, it expresses a formally necessary statement form.** Thus we can study formal necessity by studying the mathematically precise concept of validity.

4.2 Using Restricted Universes of Discourse

The preceding assumes, however, that we do not employ restricted universes of discourse when symbolizing statement forms. If we do, we can run into a problem that makes the preceding conclusion incorrect. For instance, suppose we take the universe of discourse to be the set of unicorns, which is the empty set. Letting R stand for "is red", we will symbolize the statement "There is a unicorn that is either red or not red" as $(\exists x)(Rx \vee {\sim}Rx)$. The statement "There is a unicorn that is either red or not red" is false, because there are no unicorns. But the closed formula $(\exists x)(Rx \vee {\sim}Rx)$ is valid. That is, it is true under every interpretation. This is because $(\exists x)(Rx \vee {\sim}Rx)$ is true under an interpretation if, and only if, $(Rc \vee {\sim}Rc)$ is true under some c-variant of that interpretation. But in fact, $(Rc \vee {\sim}Rc)$ will be true no matter what we assign as the denotation of c. So $(Rc \vee {\sim}Rc)$ will be true under every c-variant of the interpretation, and $(\exists x)(Rx \vee {\sim}Rx)$ will be true under the interpretation. Therefore, $(\exists x)(Rx \vee {\sim}Rx)$ is valid, but we got it by symbolizing a false statement. This difficulty arises from the fact that interpretations must have nonempty domains, but universes of discourse can be empty.

We might try avoiding this difficulty by precluding empty universes of discourse. That will guarantee that if the formula produced by symboling a statement is valid, then the statement is true. But it will not guarantee that the statement is necessary. That is because although the universe of discourse is nonempty, it may not be *logically necessary* that it is nonempty. To illustrate, suppose we symbolize the statement "There is a horse that is either a palomino or not a palomino" as $(\exists x)(Px \vee {\sim}Px)$. Because the latter formula is valid, it is true in every interpretation, and so in particular it is true in that interpretation in which the domain is the set of horses and P expresses "is a palomino". The difficulty is that it is not a necessary truth that there are horses, and if there were no horses then the statement "There is a horse that is either a palomino or not a palomino" would be false despite the validity of $(\exists x)(Px \vee {\sim}Px)$. So the validity of the formula does not ensure the necessary truth of the statement.

Although the validity of the formula does not ensure the necessary truth of a statement symbolized using a restricted universe of discourse, it *does* guarantee that *if* the universe of discourse is nonempty *then* the statement is true. So this conditional will be a necessary truth. And if we employ a universe of discourse consisting of mathematical entities, e.g., the set of all numbers, then it is a necessary truth that the universe of discourse is nonempty. That is, it is necessarily true that there are numbers. So in that case, the problem goes away. But in general, we must be more careful when using restricted universes of discourse.

We have seen that validity provides an analogue for formal necessity only at the expense of our being unable to use restricted universes of discourse in symbolizing the forms of statements. This was a very useful shortcut. There is, however, a way that it can be salvaged. Just as in the propositional calculus, the concept of validity will be used for assessing the validity of argument forms by seeing whether their corresponding conditionals are valid. Now, suppose we have an argument form that we have symbolized using a restricted universe of discourse. For example, the argument form

Everyone has a father.

Everyone who has a father has a mother.

Therefore, someone has mother.

might be symbolized as

$(\forall x)Fx$

$(\forall x)(Fx \rightarrow Mx)$

$(\exists x)Mx$

by using the class of people as a restricted universe of discourse. Suppose then that we form the corresponding conditional

$$\{[(\forall x)Fx \ \& \ (\forall x)(Fx \rightarrow Mx)] \rightarrow (\exists x)Mx\}$$

and ascertain that it is valid. It does not follow from this that the initial argument is valid. This is because should the universe of discourse be empty—should there be no people in the world—the corresponding conditional of the initial argument would be false but its symbolization is still valid. However, it does follow that another argument, which we obtain from the initial argument by adding an additional premise, is valid:

There exists at least one person.

Everyone has a father.

Everyone who has a father has a mother.

Therefore, someone has mother.

Here we have just added the premise that the universe of discourse is nonempty. Why is this new argument valid? The conditional $\{[(\forall x)Fx \ \& \ (\forall x)(Fx \rightarrow Mx)] \rightarrow (\exists x)Mx\}$ is valid, so it is true under every interpretation having a nonempty domain. Therefore, if the class of people is nonempty, then

$$\{[(\forall x)Fx \ \& \ (\forall x)(Fx \rightarrow Mx)] \rightarrow (\exists x)Mx\}$$

is true under every interpretation whose domain is the class of people. Therefore, if the class of people is nonempty, then if the two premises of the original argument are true, the conclusion will be true as well (because these statements correspond to one interpretation of the conditional). But the new premise we have added to the original argument is the premise that the class of people is nonempty. Therefore, given all three of these premises, it follows that the conclusion must be true, and so the new three-premise argument is valid.

Generally, given an argument symbolized using a restricted universe of discourse, it does not follow from the validity of the corresponding conditional that the argument is valid. But it does follow from the validity of the corresponding conditional that a new argument, which we construct by adding the additional premise that there is at least one thing in the domain, is valid. We will usually know that this new premise is true, and so we can use the argument to conclude that if the original premises are also true then the conclusion is true. However, if there is some doubt about whether the universe of discourse is nonempty, then in order to infer the consequent of the original argument from its premises, we must symbolize the argument all over again without using a restricted universe of discourse.

4.3 Related Concepts

Related to the concept of validity are three other concepts. We say that a closed formula is *invalid* if, and only if, it is not valid. And we say that a closed formula is *consistent* (often called *satisfiable*) if, and only if, its negation is invalid. A closed formula is *inconsistent* if, and only if, it is not consistent. Thus a closed formula is invalid if, and only if, it is false under at least one interpretation; a closed formula is consistent if, and only if, it is true under at least one interpretation; and a closed formula is inconsistent if, and only if, it is false under every interpretation.

4.4 Establishing Validity and Invalidity

In order to show that a closed formula is consistent it suffices to find one interpretation under which it is true, and in order to show that a closed formula is invalid it suffices to find one interpretation under which it is false. But to show that a closed formula is valid you must show that it is true under every interpretation, and in order to show that a closed formula is inconsistent you must show that it is false under every interpretation. Thus, to establish either that a closed formula is valid or that it is inconsistent, you must give some sort of general argument about all interpretations. However, to establish that a closed formula is consistent, or that it is invalid, you need merely give an example of an interpretation under which it is true, or false, respectively. It must be emphasized that you cannot establish that a closed formula is valid (or that it is inconsistent) by looking at a single interpretation, although you can establish that it is invalid (or that it is consistent) by looking at a single interpretation.

Consider some examples. Suppose we want to show that $(\forall x)(Fx \rightarrow Gx)$ is invalid. Then we must find an interpretation under which it is false. Since it is much easier to work with interpretations having small domains, let us try to find an interpretation under which $(\forall x)(Fx \rightarrow Gx)$ is false which has a one-element domain. Suppose the domain is $\{1\}$. In order to make

$(\forall x)(Fx \rightarrow Gx)$ false under some interpretation having this domain, we must make $(Fc \rightarrow Gc)$ false under some c-variant of that interpretation. But there will only be one c-variant, because there is only one object in the domain. So the denotation assigned to c must be 1. Then in order for $(Fc \rightarrow Gc)$ to be false, we must make Fc true and Gc false. Thus,1 must be a member of the extension of F, but not of G. This means that we must choose the extension {1} for F and the extension $\varnothing$ for G (because if 1 is not to be in the extension of G, then nothing can be because there is nothing else in the domain). Thus, $(\forall x)(Fx \rightarrow Gx)$ is false under the interpretation whose domain is {1} and that assigns the extension {1} to F and $\varnothing$ to G. Therefore, $(\forall x)(Fx \rightarrow Gx)$ is invalid.

This example illustrates that we can use the truth rules to guide us in searching for interpretations under which formulas have a desired truth value. However, there is often an easier way of doing it. We might try to think of meanings we can assign to the relation symbols and individual constants in a closed formula which will make it symbolize a true statement. For example, $(\forall x)(Fx \rightarrow Gx)$ means "Every F is a G". If we want to make this false we might let F stand for "is a number" and G for "is even". Then the formula symbolizes the false statement "Every number is even". Thus $(\forall x)(Fx \rightarrow Gx)$ is false under that interpretation whose domain is the set of numbers, and which assigns the set of numbers as the extension of F and the set of even numbers as the extension of G.

Next let us show that $(\forall x)(Fx \rightarrow Gx)$ is consistent. To do this we must find an interpretation under which it is true. Again let us see if we can do it with the simplest possible domain—a one-element domain, using the domain {1} again. In order for $(\forall x)(Fx \rightarrow Gx)$ to be true under an interpretation having this domain, $(Fc \rightarrow Gc)$ must be true under every c-variant of the interpretation. But once again there will only be one c-variant—that assigning 1 to c. So when the denotation of c is 1, we must have $(Fc \rightarrow Gc)$ true. There are several ways we could do this. We can either have both Fc and Gc true, or Fc false and Gc true, or both Fc and Gc false. Let us choose the first alternative. In order to make both Fc and Gc true, the extensions of both must contain 1, and their extensions cannot contain anything else, because there is nothing else in the domain. So the extensions must be simply {1}. Therefore, $(\forall x)(Fx \rightarrow Gx)$ is true under the interpretation whose domain is {1} and that assigns the extension {1} to both F and G.

Again, this problem might be done more simply by looking for meanings we can assign to F and G that make the formula express a true statement. For example, we might consider the interpretation whose domain is the set of people, and which assigns the set of bachelors as the extension of F and the set of unmarried people as the extension of G. Then $(\forall x)(Fx \rightarrow Gx)$ symbolizes the truth statement that all bachelors are unmarried, and so it is true in that corresponding interpretation.

Consider a more complex example. To show that $(\forall x)(\exists y)(Hxy \,\&\, {\sim}Hyy)$ is consistent, we must find an interpretation under which it is true. Again we can start off by seeking an interpretation having the one-element domain {1}. Then in order for $(\forall x)(\exists y)(Hxy \,\&\, {\sim}Hyy)$ to be true under an interpretation having this domain, (∃y)(Hcy & ~Hyy) must be true under every c-variant of the interpretation. But there is only one c-variant, that assigning 1 to c. So (∃y)(Hcy & ~Hyy) must be true under the c-variant of the interpretation that assigns 1 to c. But (∃y)(Hcy & ~Hyy) is true under that c-variant if, and only if, (Hcb & ~Hbb) is true under at least one b-variant of that c-variant. But again, there is only one b-variant, that assigning 1 to b. So (Hcb & ~Hbb) must be true under some interpretation that assigns 1 to both c and b. But this is impossible, because if c and b have the same denotation, then Hcb will be true if, and only if, Hbb is true (because it is the same ordered pair in both cases), and so (Hcb & ~Hbb) cannot be true. This shows that there cannot be an interpretation having a one-element domain in which (∀x)(∃y)(Hxy & ~Hyy) is true.

Let us see if we can find an interpretation having a two-element domain, {1,2}, in which (∀x)(∃y)(Hxy & ~Hyy) is true. This closed formula will be true under an interpretation having this domain if, and only if, (Hcb & ~Hbb) is true under some b-variant of each c-variant of the interpretation. There are two c-variants—those assigning 1 and 2 to c—and there are two b-variants of each c-variant—those assigning 1 and 2 to b. Is there an extension we can assign to H that will give this result? First consider the c-variant assigning the denotation 1 to c. Under some b-variant of this c-variant (Hcb & ~Hbb) must be true. For (Hcb & ~Hbb) to be true we must have Hcb true and Hbb false. To have Hcb true under some b-variant of c-variant, there must be some ordered pair in the extension of H whose first element is 1. The second element must either be 1 or 2, depending upon which denotation b has. If we allow Hcb to be true under the b-variant that assigns 1 to b, then ⟨1,1⟩ will be a member of the extension of H. But then Hbb will also be true, and (Hcb & ~Hbb) will then be false; so this b-variant will not do the trick. Suppose then that we let Hcb be true under the other b-variant—that assigning 2 to b. Then ⟨1,2⟩ must be in the extension of H. Also, in order to have Hbb false, ⟨2,2⟩ must not be in the extension of H. Thus, by requiring that ⟨1,2⟩ is in the extension of H, and ⟨2,2⟩ is not in the extension of H, we can make (Hcb & ~Hbb) true in some b-variant of the first c-variant. Now let us see if we can also make it true in the second c-variant. Here we assign 2 to c. Again, we want to make (Hcb & ~Hbb) true under some b-variant. Let us see whether we can make it true under the b-variant that assigns 1 to b. In order to make Hcb true and Hbb false under that b-variant of this c-variant, we must have ⟨2,1⟩ in the extension of H, and we must not have ⟨1,1⟩ in the extension of H. Therefore, we see that we can make (∀x)(∃y)(Hxy & ~Hyy) true under an interpretation whose domain is {1,2} if we can assign an extension to H which contains ⟨1,2⟩ and ⟨2,1⟩, but does not contain ⟨1,1⟩ or ⟨2,2⟩. We can do this simply by assigning the extension {⟨1,2⟩, ⟨2,1⟩}. Therefore, (∀x)(∃y)(Hxy & ~Hyy) is consistent.

We can also attempt to solve this problem by looking for a restricted universe of discourse and a meaning for H that will make the formula express a true statement. If we let the universe of discourse be the set of positive integers, and we let H mean "(1) is less than (2)", then (∀x)(∃y)(Hxy & ~Hyy) means "For every positive integer, there is a positive integer greater than it but not greater than itself". Thus $(\forall x)(\exists y)(Hxy \,\&\, {\sim}Hyy)$ is true under that interpretation whose domain is the set of positive integers, and which assigns to H the extension $\{\langle x, y\rangle \mid x \text{ is less than } y\}$. Another meaning we might assign to H to make $(\forall x)(\exists y)(Hxy \,\&\, {\sim}Hyy)$ true is "(1) is not equal to (2)". Then $(\forall x)(\exists y)(Hxy \,\&\, {\sim}Hyy)$ means "Given any positive integer, there is a positive integer that is unequal to it but not unequal to itself". Notice that if we let H mean this, and we let our universe of discourse be {1,2}, we arrive at the same interpretation as we did in the preceding paragraph merely by considering the truth rules. However, it may be hard to find such intuitive interpretations for complex formulas. Using the truth rules for guidance is often a better way to find interpretations making complex formulas true.

It is generally much easier to work with interpretations having small domains, so when trying to find an interpretation under which a closed formula is false, it is generally wise to begin with the smallest possible domain, a one-element domain. If this does not work, then try a two-element domain, and so on. Occasionally, however, a closed formula will only be true under an interpretation having an infinite domain. The closed formula

$$\{(\forall x)(\forall y)(\forall z)[(Rxy \,\&\, Ryz) \rightarrow Rxz] \,\&\, [(\forall x){\sim}Rxx \,\&\, (\forall x)(\exists y)Rxy]\}$$

can be shown to be false under any interpretation having a finite domain. But it is consistent, because it can be made true in an interpretation having an infinite domain. If we let the domain be the set of all positive integers, and let R be the relation "(1) is a positive integer less than (2)", this closed formula is true. So if one cannot find an interpretation having a small domain under which a closed formula is true, it does not automatically follow that the closed formula is inconsistent. You might have to try an infinite domain in order to make the closed formula true.

We have seen that we can show a closed formula to be consistent or invalid by finding a single interpretation under which it is true, or false, respectively. But it is more difficult to show that a closed formula is inconsistent or valid. In the propositional calculus, truth tables provide a mechanical procedure for determining whether a formula is valid (i.e., a tautology). Truth tables can get long, but they are still finite structures that in principle survey all of the possible assignments of truth values to the atomic parts of the formula. There is no similar mechanical procedure that can be used for testing validity in the predicate calculus. There are always infinitely many ways of interpreting the quantifiers, relation symbols, and individual constants in a formula of the predicate calculus. There is no way to survey infinitely many interpretations, so there can be nothing like truth tables for the predicate calculus. Instead, we must give some sort of general argument to show that the closed formula is either true under all interpretations (valid), or else false under all interpretations (inconsistent), depending upon what we are trying to show. Let us look at some examples of this.

Suppose first that we want to establish the validity of $(\forall x)(Px \rightarrow Px)$. This closed formula is valid if, and only if, it is true under every interpretation.

So consider an arbitrary interpretation. $(\forall x)(Px \rightarrow Px)$ is true under that interpretation if, and only if, $(Pc \rightarrow Pc)$ is true under every c-variant of that interpretation. Just as in the propositional calculus, $(Pc \rightarrow Pc)$ is true under every interpretation. A c-variant of an interpretation is an interpretation, so it follows that $(Pc \rightarrow Pc)$ is true under every c-variant of any interpretation. Therefore, $(\forall x)(Px \rightarrow Px)$ must be true under every interpretation; that is, $(\forall x)(Px \rightarrow Px)$ must be valid.

Next suppose we want to show that $(\exists y)(Fy \,\&\, {\sim}Fy)$ is inconsistent — that is, it is false under every interpretation. Again, consider an arbitrary interpretation. $(\exists y)(Fy \,\&\, {\sim}Fy)$ will be true under that interpretation if, and only if, $(Fc \,\&\, {\sim}Fc)$ is true under some c-variant of that interpretation. But in order for $(Fc \,\&\, {\sim}Fc)$ to be true, Fc would have to be both true and false, which is impossible. Thus, $(Fc \,\&\, {\sim}Fc)$ is not true under any c-variant of the interpretation, and $(\exists y)(Fy \,\&\, {\sim}Fy)$ is false under the interpretation. This is true for any interpretation we might choose, so $(\exists y)(Fy \,\&\, {\sim}Fy)$ will be false under every interpretation, and so inconsistent.

To take a more complicated case, let us show that $(\exists x)(Fx \rightarrow (\forall y)Fy)$ is valid. Consider any interpretation. $(\exists x)(Fx \rightarrow (\forall y)Fy)$ will be true under that interpretation if, and only if, $(Fc \rightarrow (\forall y)Fy)$ is true under some c-variant of that interpretation. $(Fc \rightarrow (\forall y)Fy)$ will be true under a given c-variant as long as we do not have Fc true and $(\forall y)Fy$ false. $(\forall y)Fy$ does not contain c, so it is true under the c-variant if, and only if, it is true under the original interpretation. Now we have two cases to consider:

Case 1. Suppose $(\forall y)Fy$ is true under the original interpretation. Then $(Fc \rightarrow (\forall y)Fy)$ will be true under every c-variant of the original interpretation (because it has a true consequent), and so $(\exists x)(Fx \rightarrow (\forall y)Fy)$ will be true under the original interpretation.

Case 2. Suppose $(\forall y)Fy$ is false under the original interpretation, then Fb must be false under some b-variant of the original interpretation; that is, there must be something we can assign as the denotation of b that will make Fb false. But now, suppose we consider the c-variant that assigns the same denotation to c as it does to b. Then Fc will also be false. But then $(Fc \rightarrow (\forall y)Fy)$, being a conditional with a false antecedent, will be true under that c-variant. Thus $(\exists x)(Fx \rightarrow (\forall y)Fy)$ will be true under the original interpretation. In either case, $(\exists x)(Fx \rightarrow (\forall y)Fy)$ will be true under the original interpretation. This is true for every interpretation, so $(\exists x)(Fx \rightarrow (\forall y)Fy)$ must be true under every interpretation—it is valid.

It is often quite difficult to give an argument to show that a closed formula is valid. Clearly it would be advantageous to have some sort of general rules to help us along, like the rule of tautological implication in the propositional calculus. In the next two sections we will be concerned with finding such rules, and then we will use these rules to construct derivations in the predicate calculus much like derivations in the propositional calculus.

Exercises

A. For each of the following closed formulas, find an interpretation under which it is true:

1. Fc
2. (Fa & ~Fb)
3. $(\forall x)$Fx
4. $(\exists x)$Fxa
5. $(\forall x)(\exists y)$Hxy
6. $(\exists y)(\forall x)$Hxy
7. $(\forall x)(Fx \rightarrow Gx)$
8. $(\forall x)(Fxx \rightarrow (\exists y)Fxy)$

B. Recalling that every formula of the propositional calculus is a closed formula of the predicate calculus, show that if a formula of the propositional calculus is truth-functionally consistent, then it is a consistent formula of the predicate calculus.

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5. Implication and Equivalence

In the propositional calculus, the concepts of tautological implication and tautological equivalence were defined in terms of tautologicity. In the predicate calculus we can define analogous concepts of implication and equivalence in terms of the concept of validity. Given two closed formulas A and B of the predicate calculus, we say that A *implies* B if, and only if, the conditional $(A \rightarrow B)$ is valid. So, for example, $(\forall x)$Fx implies Fa, because the conditional $[(\forall x)Fx \rightarrow Fa]$ is valid. More generally:

> A set of formulas $A_1,\ldots,A_n$ *implies* a formula B if, and only if, $\vdash [(A_1\&\ldots\&A_n) \rightarrow B]$.

We abbreviate "$A_1,\ldots,A_n$ implies B" as "$A_1,\ldots,A_n \vdash B$". Because all tautologies are valid, it follows that if a closed formula or set of closed formulas tautologically implies another closed formula, then it implies the other closed formula. That is, tautological implication is a special case of implication.

To say that a conditional is true under an interpretation means that if the interpretation makes the antecedent true it also makes the consequent true. Thus to say that a conditional is valid means that every interpretation that makes the antecedent true also makes the consequent true. Thus, $A_1,\ldots,A_n \vdash B$ if, and only if, every interpretation making $A_1,\ldots,A_n$ all true also makes B true.

We define equivalence in an analogous manner. We say that two closed formulas A and B are *equivalent* if, and only if, they are true relative to the same interpretations. This is the same as saying that each formula implies the other. Equivalently, A and B are equivalent if, and only if, $(A \leftrightarrow B)$ is valid. Thus, for example, $(\forall x)(Fx \rightarrow Gx)$ is equivalent to $(\forall x)(\sim Gx \rightarrow \sim Fx)$, because $[(\forall x)(Fx \rightarrow Gx) \leftrightarrow (\forall x)(\sim Gx \rightarrow \sim Fx)]$ is valid.

All of the implications and equivalences employed in the propositional calculus still hold in the predicate calculus. There are also a few new ones that will be particularly useful. In addition to the tautological implications I1–I15, we have:

I16. $(\forall x)A \vdash \mathrm{Sb}(c/x)A$

I17. $\mathrm{Sb}(c/x)A \vdash (\exists x)A$

Roughly, I16 says that if A is true of everything, then it is true of c. I17 says that if A is true of c, then it is true of something.

In addition to the tautological equivalences E1 – E20, we have the following equivalences:

E21. $\left\{\begin{array}{l} (\forall x)A \text{ eq. } \sim(\exists x)\sim A \\ (\exists x)A \text{ eq. } \sim(\forall x)\sim A \\ \sim(\forall x)A \text{ eq. } (\exists x)\sim A \\ \sim(\exists x)A \text{ eq. } (\forall x)\sim A \end{array}\right.$

$$\left.\begin{array}{ll} \text{E22.} & (\forall x)(A \vee B) \text{ eq. } [(\forall x)A \vee B] \\ \text{E23.} & (B \rightarrow (\forall x)A) \text{ eq. } (\forall x)(B \rightarrow A) \\ \text{E24.} & ((\exists x)A \rightarrow B) \text{ eq. } (\forall x)(A \rightarrow B) \end{array}\right\} \text{provided there are no free occurrences of } x \text{ in } B.$$

$$\left.\begin{array}{ll} \text{E25.} & (\forall x)A \text{ eq. } (\forall y)\mathrm{Sb}(y/x)A \\ \text{E26.} & (\exists x)A \text{ eq. } (\exists y)\mathrm{Sb}(y/x)A \end{array}\right\} \text{provided } y \text{ does not occur in } A.$$

E23 and E24 are somewhat surprising, so let us see why they are true. First consider E23. By E4, $(B \rightarrow (\forall x)A)$ is equivalent to $(\sim B \vee (\forall x)A)$. By E12, the latter is equivalent to $((\forall x)A \vee \sim B)$, and by E22 this is equivalent to $(\forall x)(A \vee \sim B)$. $(\forall x)(A \vee \sim B)$ is true under a given interpretation if, and only if, $\mathrm{Sb}(c/x)(A \vee \sim B)$ is true under every c-variant of that interpretation (where c is some individual constant not occurring in $(A \vee \sim B)$). By E12, $\mathrm{Sb}(c/x)(A \vee \sim B)$ is equivalent to $\mathrm{Sb}(c/x)(\sim B \vee A)$, and then by E4 this is equivalent to $\mathrm{Sb}(c/x)(B \rightarrow A)$. Thus $(\forall x)(A \vee \sim B)$ is true under an interpretation if, and only if, $\mathrm{Sb}(c/x)(B \rightarrow A)$ is true under every c-variant of that interpretation. But the latter is just the condition under which $(\forall x)(B \rightarrow A)$ is true. Therefore, $(B \rightarrow (\forall x)A)$ is equivalent to $(\forall x)(B \rightarrow A)$.

Similarly, $((\exists x)A \rightarrow B)$ is equivalent to $(\sim(\exists x)A \vee B)$, by E4. By E21 this is equivalent to $((\forall x)\sim A \vee B)$, which by E22 is equivalent to $(\forall x)(\sim A \vee B)$. By reasoning as above and using E4, this can be seen to be equivalent to $(\forall x)(A \rightarrow B)$.

It should be emphasized that in using E22 – E24, B cannot contain any free occurrences of x. For example, if B did contain free occurrences of x, then $((\forall x)A \vee B)$ would contain free occurrences of x and so would not even be a closed formula.

E25 and E26 result from the fact that, for example, $(\forall x)Fx$ and $(\forall y)Fy$ mean the same thing. They both say simply that everything is an F. In terms of the truth rules, they are both true under an interpretation if, and only if, Fc is true under every c-variant of the interpretation. Thus, it makes no difference which bound variable we use. However, the restriction that y

does not occur in A is essential. Without it we would find ourselves saying, for example, that (∀x)(∀y)Hxy is equivalent to (∀y)(∀y)Hyy. These closed formulas, however, are not equivalent. The first says that given any two things, they stand in the relation H to one another, whereas the second says that everything stands in the relation H to itself (the first quantifier is a vacuous quantifier and does not affect the meaning).

Exercises

Prove the following metatheorems. The proofs are similar to the proofs given in the propositional calculus.

1. Show that if a closed formula is implied by a valid closed formula, then it is valid.
2. Show that implication is strongly transitive.
3. Show that implication is adjunctive.
4. Using the result of 2, show that if A, B, and C are closed formulas, and $A \vdash B$, $B \vdash C$, and $C \vdash A$, then A, B, and C are all equivalent.

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6. Universal Generalization

A principle of fundamental importance in the predicate calculus is the principle of *universal generalization*:

> ***Metatheorem***: If A is any formula of the predicate calculus that contains free occurrences of some variable x but not of any other variable, then given any individual constant c not occurring in A, if Sb(c/x)A is valid, then (∀x)A is valid.

For example, consider the formula (Fx ∨ ~Fx). The closed formula (Fc ∨ ~Fc) is valid, because it is a tautology. Therefore, by the above principle, (∀x)(Fx ∨ ~Fx) is valid. This principle is proven as follows. Suppose Sb(c/x)A is valid. This means that Sb(c/x)A is true under every interpretation. In particular, given any one interpretation, Sb(c/x)A is not only true under that interpretation, but also under every other interpretation exactly like that interpretation except that it assigns something different as the denotation of c. In other words, given any interpretation, Sb(c/x)A is true under every c-variant of that interpretation. Therefore, given any interpretation, (∀x)A is true under that interpretation. Thus, (∀x)A is valid. ■

We use the principle of universal generalization to establish that universal generalizations are valid when we know that instances of the scope of the quantifier are valid. This is particularly useful when taken together with the observation that all tautologies are valid. Given any tautology, we can construct a valid universal generalization by replacing individual constants by variables and appending universal quantifiers. For example, the closed formula

$$[(\mathrm{Fa} \leftrightarrow \sim\mathrm{Gb}) \leftrightarrow \sim(\mathrm{Fa} \leftrightarrow \mathrm{Gb})]$$

is a tautology. From this, by universal generalization, we can conclude that

$$(\forall y)[(\mathrm{Fa} \leftrightarrow \sim\mathrm{Gy}) \leftrightarrow \sim(\mathrm{Fa} \leftrightarrow \mathrm{Gy})]$$

is valid. Then by a second application of universal generalization, we can conclude that

$(\forall x)(\forall y)[(\mathrm{Fx} \leftrightarrow \sim\mathrm{Gy}) \leftrightarrow \sim(\mathrm{Fx} \leftrightarrow \mathrm{Gy})]$ is valid.

We can construct an analogous rule of universal generalization for implications:

> ***Metatheorem***: Suppose A is any formula of the predicate calculus that contains free occurrences of some variable x but not of any other variable, and suppose that c is some individual constant that does not occur in A. Then if $B_1,\ldots,B_n$ are closed formulas none of which contain c, and $B_1,\ldots,B_n \vdash \mathrm{Sb}(c/x)A$, it is also true that $B_1,\ldots,B_n \vdash (\forall x)A$.

For example, the set of closed formulas $(\forall x)(\mathrm{Fx} \rightarrow \mathrm{Gx})$, $(\forall x)(\mathrm{Gx} \rightarrow \mathrm{Hx})$ implies the closed formula $(\mathrm{Fc} \rightarrow \mathrm{Hc})$. Neither $(\forall x)(\mathrm{Fx} \rightarrow \mathrm{Gx})$ nor $(\forall x)(\mathrm{Gx} \rightarrow \mathrm{Hx})$ contains c. Therefore, this set of closed formulas also implies $(\forall x)(\mathrm{Fx} \rightarrow \mathrm{Hx})$.

This principle is proven as follows. Suppose $B_1,\ldots,B_n$ are closed formulas none of which contain c, and $B_1,\ldots,B_n \vdash \mathrm{Sb}(c/x)A$, where A does not contain c either. By definition, this means that $[(B_1 \,\&\, \ldots \,\&\, B_n) \rightarrow \mathrm{Sb}(c/x)A]$ is valid. As $B_1,\ldots,B_n$ are closed formulas, they contain no free occurrences of x, so

$$\mathrm{Sb}(c/x)[(B_1 \,\&\, \ldots \,\&\, B_n) \rightarrow A]$$

is the same closed formula as $[(B_1 \,\&\, \ldots \,\&\, B_n)) \rightarrow \mathrm{Sb}(c/x)A]$. Thus

$$\mathrm{Sb}(c/x)[(B_1 \,\&\, \ldots \,\&\, B_n) \rightarrow A]$$

is valid. Furthermore, as c does not occur in any of $B_1,\ldots,B_n$ or A, c does not occur in $[(B_1 \,\&\, \ldots \,\&\, B_n) \rightarrow A]$. Thus, by the first principle of universal generalization, $(\forall x)[(B_1 \,\&\, \ldots \,\&\, B_n) \rightarrow A]$ is valid. x does not occur free in $(B_1 \,\&\, \ldots \,\&\, B_n)$, so by E23, $(\forall x)[(B_1 \,\&\, \ldots \,\&\, B_n) \rightarrow A]$ is equivalent to $[(B_1 \,\&\, \ldots \,\&\, B_n) \rightarrow (\forall x)A]$, and consequently this latter closed formula is also valid. Therefore, the set of closed formulas $B_1,\ldots,B_n$ implies $(\forall x)A$. ■

The principle of universal generalization corresponds to a frequently used pattern of informal reasoning. Suppose we want to prove that something is true of all objects in a certain domain, say the integers. In order to prove that it is true of all integers, we might reason as follows. First we pick some arbitrary integer, and show that it is true of that integer. Then we observe that our proof that it is true of that integer does not assume anything about that integer which is not true of all integers, and so the same proof will work for any other integer as well. Thus we conclude that it is true of every integer. Observing that the proof does not assume anything about the integer we chose that is not true of all integers is the same as observing that the premises of our proof do not refer to that integer—that they do not contain *c*. So we are just using universal generalization.

Exercise

Using universal generalization, show that $(\forall x)(Fx \rightarrow Gx), (\forall x)Fx \vdash (\forall x)Gx$.
