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Derivations in the Predicate Calculus

1. Rules of Inference

Derivations in the predicate calculus have essentially the same structure as derivations in the propositional calculus. In particular, we continue to employ the inference rules P (premise introduction), I (implication), C (conditionalization), DN (double negation), and R (reductio ad absurdum). We have added two implications and six equivalences to the list of implications and equivalences used by rule I, but the rule is conceptually unchanged. In addition, we will add one new inference rule. This will be based upon the principle of universal generalization discussed in chapter six. That rule was formulated as follows:

Suppose \( A \) is any formula of the predicate calculus that contains free occurrences of some variable \( x \) but not of any other variable, and suppose that \( c \) is some individual constant that does not occur in \( A \). Then if \( B_1, \ldots, B_n \) are closed formulas none of which contain \( c \), and \( B_1, \ldots, B_n \vdash \text{Sb}(c/x)A \), it is also true that \( B_1, \ldots, B_n \vdash (\forall x)A \).

Recall that lines of a derivation express implications. They tell us that the premises of the line imply the conclusion drawn on that line (or if a line has no premises, they tell us that the conclusion is valid). Accordingly, we can employ universal generalization to reason as follows:

\[ \text{RULE UG: Universal Generalization} \quad \text{Suppose } A \text{ is a formula of the predicate calculus, } \mathbf{ x } \text{ is a variable, and } A \text{ does not contain free occurrences of any variable other than } \mathbf{ x }, \text{ and suppose } c \text{ is some individual constant that does not occur in } A. \text{ Then if } \text{Sb}(c/x)A \text{ appears on some line of a derivation, and it either has no premises or none of its premises contain } c, \text{ we can write } (\forall x)A \text{ on any later line of the derivation, taking for premise numbers all the premise numbers of } \text{Sb}(c/x)A. \]

We will later add an additional rule of inference to make it easier to construct derivations in the predicate calculus, but for now we will take derivations to be defined by the rules P, I, C, DN, R, and UG. That is, a derivation in the predicate calculus is any sequence of lines that can be constructed in accordance with these six rules of inference.

Now let us look at some sample derivations without, for the moment, worrying about the strategies involved in constructing them. Suppose we want to construct a derivation of \( (\forall x)[Fx \to (Fx \lor Gx)] \). We might proceed as follows:
To take another example, suppose we want to derive $(\forall x)(\exists y)Fxy$ from $(\exists y)(\forall x)Fxy$. We might proceed as follows:

\[(1) \quad 1. \quad (\exists y)(\forall x)Fxy \quad P \\
(2) \quad 2. \quad (\forall x)Fxa \quad P \\
(2) \quad 3. \quad Fba \quad (I^{16}), 2 \\
(2) \quad 4. \quad (\exists y)Fby \quad (I^{17}), 3 \\
(2) \quad 5. \quad (\forall x)(\exists y)Fxy \quad UG, 4 \\
\quad 6. \quad [(\forall x)Fxa \rightarrow (\forall x)(\exists y)Fxy] \quad C, 2, 5 \\
\quad 7. \quad (\forall y) [(\forall x)Fxy \rightarrow (\forall x)(\exists y)Fxy] \quad UG, 6 \\
\quad 8. \quad [(\exists y)(\forall x)Fxy \rightarrow (\forall x)(\exists y)Fxy] \quad (E^{24}), 7 \\
(1) \quad 9. \quad (\forall x)(\exists y)Fxy \quad (I^{9}), 1, 8 \]

Suppose next we want to derive $(\exists y)Gy$ from $(\forall x)(Fx \rightarrow Gx)$ and $(\exists z)Fz$. We might proceed as follows:

\[(1) \quad 1. \quad (\forall x)(Fx \rightarrow Gx)P \\
(2) \quad 2. \quad (\exists z)Fz \quad P \\
(3) \quad 3. \quad Fa \quad P \\
(1) \quad 4. \quad (Fa \rightarrow Ga) \quad (I^{16}), 1 \\
(1, 3) \quad 5. \quad Ga \quad (I^{9}), 3, 4 \\
(1, 3) \quad 6. \quad (\exists y)Gy \quad (I^{17}), 5 \\
\quad 7. \quad [Fa \rightarrow (\exists y)Gy] \quad C, 3, 6 \\
(1) \quad 8. \quad (\forall z)[Fz \rightarrow (\exists y)Gy] \quad UG, 7 \\
(1) \quad 9. \quad [(\exists z)Fz \rightarrow (\exists y)Gy] \quad (E^{24}), 8 \\
(1, 2) \quad 10. \quad (\exists y)Gy \quad (I^{9}), 2, 9 \]

**Exercises**

In each of the following purported derivations, check the lines for errors, and indicate the numbers of the lines on which errors occur. An error is said to occur in a line if the line cannot be introduced in accordance with the rules of inference. This means that in checking a line for errors, you only look at that line to see whether it has the necessary relation to the previous lines, and treat the previous lines as if they were all correct. If a line is incorrect, try to find a way of correcting it (perhaps by adding additional lines).
### Derivations in the Predicate Calculus

1. \( (\forall x)(Fx \lor Gx) \) P

2. \( (\exists x)(\forall y)(Fx \to Gy) \) P

3. \( (\exists x)\neg Gx \) P

4. \( (\forall y)(Fa \to Gy) \) (I17), 2

5. \( (Fa \to Gb) \) (I16), 4

6. \( \neg Gb \) P

7. \( \neg Fa \) (I10), 5, 6

8. \( (Fa \lor Ga) \) (I16), 1

9. \( Ga \) (I11), 7, 8

10. \( (\forall x)Gx \) UG, 9

11. \[ (\exists x)(Gx \to (\forall y)Hxy) \to (\forall x)(\exists y)(Gy \to Hyx) \] C, 2, 8

12. \( (\forall x)(\exists y)(Gy \to Hyx) \to (\forall x)(\exists y)(Gy \to Hyx) \) (I17), 9

13. \( (\forall x)(\exists y)(Gy \to Hyx) \) (I9), 1, 10

### 2. Strategies

Just as in the propositional calculus, there are strategies that can guide one in the construction of derivations. The strategies for conjunctions, disjunctions, conditionals, and biconditionals are the same as before. The strategy for negations must be augmented by the observation that, by E21, \( \neg(\forall x)P \) is equivalent to \( (\exists x)\neg P \), and \( \neg(\exists x)P \) is equivalent to \( (\forall x)\neg P \). To these we add strategies for universal generalizations and existential generalizations.

#### 2.1 Universal Generalizations

When reasoning forward, we may derive conclusions from a universal generalization, and when reasoning backward we may derive new interests.
from interests in universal generalizations. We have separate strategies for each.

**Forward reasoning**

Typically, when reasoning forward from a universal generalization we use I16 to obtain one or more instances of the generalization, and then continue our reasoning from the instances. This is called *universal instantiation*. For example, suppose we want to derive \((F_a \& G_a)\) from \((\forall x)Fx\). We might proceed as follows:

1. \((\forall x)Fx\) \(P\)
2. \(F_a\) \((I16), 1\)
3. \(F_b\) \((I16), 1\)
4. \((F_a \& F_b)\) \((I14), 1\)

Note that we can use universal instantiation to infer as many instances of the universal generalization as may be useful. The only difficult part of this strategy is figuring out which instances will be useful, and even that is usually not very difficult.

**Backward reasoning**

Backward reasoning consists of adopting interest in other conclusions from which we could infer the formula we are trying to derive. If our objective is to derive a universal generalization, we will usually do this by employing UG. For instance, suppose we want to derive \((\forall x)(\neg G_x \rightarrow \neg F_x)\) from \((\forall x)(F_x \rightarrow G_x)\). Then we construct the following annotated derivation:

1. \((\forall x)(F_x \rightarrow G_x)\) \(P\)
2. \((\neg G_a \rightarrow \neg F_a)\) for 1 by \((UG)\)
3. \((\neg G_a \rightarrow \neg F_a)\) \((E16), 2\) \(this\ discharges\ interest\ 2\)
4. \((\forall x)(\neg G_x \rightarrow \neg F_x)\) \((UG), 3\) \(this\ discharges\ interest\ 1\)

The general strategy is to adopt interest in an instance of the generalization we are trying to derive. In order to use universal generalization later, the individual constant employed in the instance must be a “new” constant not occurring elsewhere in the formula or in any premises of the formula. So given an interest in a formula \((\forall x)A\), we choose an individual constant \(c\) that does not occur in \(A\) or in any of the premises, and adopt interest in \(Sb(c/x)A\). If we are able to derive the latter, we can then derive \((\forall x)A\) by universal generalization.
Exercises

For each of the following, construct an annotated derivation of the closed formula below the line from the closed formulas above the line. Keep the strategies in mind.

1. \( (\forall x)[Fx \to (Hx & \neg Gx)] \)
   \( (Ga \to \neg Fa) \)

2. \( (\forall x)Fx \)
   \( (\forall x)(Fx \to Gx) \)
   \( (\forall x)Gx \)

2.2 Existential Generalizations

We also have distinct strategies for reasoning forward or backward with existential generalizations.

Backward reasoning

To derive an existential generalization from something else, we typically use EI7, deriving the existential generalization from an instance of it. For example, if we want to derive \( (\exists x)Fx \) from \( (\forall x)Fx \), we might proceed as follows:

(1) 1. \( (\forall x)Fx \) P
(1) 1. \( (\exists x)Fx \)
(2) 2. \( Fa \) for 1 by (EI7)
(1) 2. \( Fa \) (EI6), 1 this discharges interest 2
(1) 3. \( (\exists x)Fx \) (EI7), 2 this discharges interest 1

Forward reasoning

Our most complex strategy concerns how to reason forward from an existential generalization. Before considering the general strategy, let us look at an example of it. To derive \( (\exists x)Gx \) from \( (\exists x)(Fx & Gx) \). We might proceed as follows:

(1) 1. \( (\exists x)(Fx & Gx) \) P
(1) 1. \( (\exists x)Gx \)
(2) 2. \( (Fa & Ga) \) P
(2) 2. \( (\exists x)Gx \) for 1, by existential instantiation
(2) 3. \( Ga \) for 2 by (EI7)
(2) 3. \( Ga \) (EI2), 2 this discharges interest 3
(2) 4. \( (\exists x)Gx \) (EI7), 3 this discharges interest 2
5. \[ [(Fa \& Ga) \rightarrow (\exists x)Gx] \] C, 2, 4

6. \[ (\forall x)[(Fx \& Gx) \rightarrow (\exists x)Gx] \] UG, 5

7. \[ [(\exists x)(Fx \& Gx) \rightarrow (\exists x)Gx] \] (E24), 6

8. \[ (\exists x)Gx \] (I9), 1, 7 \textit{this discharges interest 1}

This strategy is called \textit{existential instantiation}. In order to derive something from \((\exists x)(Fx \& Gx)\), we took as a new premise \((Fa \& Ga)\), which is \(Sb(a/x)(Fx \& Gx)\). Then we proceeded to derive our desired conclusion \((\exists x)Gx\) from this new premise. Next we conditionalized, obtaining

\[ [(Fa \& Ga) \rightarrow (\exists x)Gx] \]

Then we used UG, to obtain \((\forall x)[(Fx \& Gx) \rightarrow (\exists x)Gx]\). Because \(x\) does not occur free in \((\exists x)Gx\), we were able to use E24 to convert the universal generalization into \([(\exists x)(Fx \& Gx) \rightarrow (\exists x)Gx]\), and finally, by modus ponens, we inferred our desired conclusion from the original premises.

Generalizing this strategy, suppose we want to derive some closed formula \(B\) from the premise \((\exists x)A\) together with some other premises (whose numbers are indicated in the following derivation by the dots in the left-hand column). We choose an individual constant \(c\), that does not occur in \(B\) or \(A\) or any of the premises, and proceed as follows:

\[
\begin{align*}
(1) & \quad 1. \quad \ldots \quad P \\
(2) & \quad 2. \quad \ldots \quad P \\
& \quad \vdots \\
(n) & \quad n. \quad (\exists x)A \\
& \quad \vdots \\
\quad (1, \ldots, n) & \quad 2. \quad B \quad \text{for 1 by existential instantiation} \\
(n + 1) & \quad n + 1. \quad Sb(c/x)A \\
& \quad \vdots \\
\quad (n + 1, \ldots) & \quad k. \quad B \quad \text{this discharges interest 2} \\
\quad \vdots & \quad \ldots \quad k + 1. \quad [Sb(c/x)A \rightarrow B] \\
\quad \vdots & \quad \ldots \quad k + 2. \quad (\forall x)(A \rightarrow B) \\
\quad \vdots & \quad \ldots \quad k + 3. \quad [(\exists x)A \rightarrow B] \\
(n, \ldots) & \quad k + 4. \quad B \quad (I9), n, k + 3
\end{align*}
\]

Notice that the second and third derivation on page 144 both use existential instantiation.

The intuitive rationale for existential instantiation is the following. Suppose we know that \((\exists x)Fx\) is true, i.e., that something is an \(F\). From this we want to derive a conclusion \(Q\). We might begin our argument by saying, “We know that something is an \(F\); call it ‘\(c\)’.” This is the same as
taking \( Fc \) as a new premise. If we can then derive \( Q \) from \( Fc \) without assuming anything about \( c \), then \( Q \) will follow from \( Fc \) regardless of what \( c \) is, and so \( Q \) will follow from the mere fact that there is something that is an \( F \). This reasoning corresponds to the last three steps of existential instantiation.

Notice that in order for us to be able to use UG on line \((k + 2)\), \( c \) cannot occur in \( A \) or \( B \) or any of the premises. This corresponds in the intuitive reasoning pattern to requiring that we have not assumed anything about \( c \). It is important to keep this restriction in mind, because it is a very common mistake to overlook it.

**Exercises**

For each of the following, construct an annotated derivation of the closed formula below the line from the closed formulas above the line. Keep the strategies in mind.

1. \( Fa \)

   \[
   (\exists x)(Fx \lor Gx)
   \]

2. \( (\exists x)Fx \)

   \[
   (\exists x)(Fx \lor Gx)
   \]

3. \( (\exists x)(Fx \lor Gx) \)

   \[
   \neg(\exists x)Fx
   \]

   \[
   (\exists x)Gx
   \]

**2.3 A General Strategy**

These strategies combine to give us one basic strategy for derivations in the predicate calculus: eliminate quantifiers from the premises using existential and universal instantiation, eliminate quantifiers from the interests using existential and universal generalization, then proceed as in the propositional calculus to derive the conclusion without quantifiers, and finally replace quantifiers using existential and universal generalization. Let us look at some examples of this.

To illustrate, suppose we want to derive \( (\exists x)(\forall z)(\exists y)Fxyz \) from \( (\forall x)(\exists y)(\forall z)Fxyz \). We might proceed as follows:

\[
\begin{align*}
(1) & \quad 1. \quad (\forall x)(\exists y)(\forall z)Fxyz & \text{P} \\
(1) & \quad 1. \quad (\exists x)(\forall z)(\exists y)Fxyz \\
(1) & \quad 2. \quad (\exists y)(\forall z)Fxyz & (I16), \ 1
\end{align*}
\]
(3) 3. $(\forall z)Fabz$ P (beginning existential instantiation)
(3) 2. $(\exists x)(\forall z)(\exists y)Fxyz$ for 1, by existential instantiation
(3) 3. $(\forall z)(\exists y)Fayz$ for 2 by (I17)
(3) 4. $(\exists y)Fayc$ for 3 by UG
(3) 5. $Fabc$ for 4 by (I17)
(3) 4. $Fabc$ (I16), 3 this discharges interest 5
(3) 5. $(\exists y)Fayc$ (I17), 4 this discharges interest 4
(3) 6. $(\forall z)(\exists y)Fayz$ UG, 5 this discharges interest 3
(3) 7. $(\exists x)(\forall z)(\exists y)Fxyz$ (I17), 6 this discharges interest 2
8. $[(\forall z)Fabz \rightarrow (\exists x)(\forall z)(\exists y)Fxyz]$ C, 3, 7
9. $(\forall y)[(\forall z)Fayz \rightarrow (\exists x)(\forall z)(\exists y)Fxyz]$ UG, 8
10. $[(\exists y)(\forall z)Fayz \rightarrow (\exists x)(\forall z)(\exists y)Fxyz]$ (E24), 9
(1) 11. $(\exists x)(\forall z)(\exists y)Fxyz$ (I9), 2, 10 (completing existential instantiation) this discharges interest 1

Here we began by eliminating quantifiers in conclusions 1 – 4 and interests 1 – 5. This gave us our conclusion without quantifiers. Then in steps 5 – 11 we replaced the quantifiers to give us our conclusion with the desired quantifiers.

To take a more difficult example, suppose we want to derive $(\forall x)(Fx \rightarrow \neg Px)$ from the set of premises $(\forall x)[Gx \rightarrow (\forall y)(Fy \rightarrow Hxy)]$, $(\forall x)[Gx \rightarrow (\forall z)(Pz \rightarrow \neg Hxz)]$, and $(\exists x)Gx$:

(1) 1. $(\forall x)[Gx \rightarrow (\forall y)(Fy \rightarrow Hxy)]$ P
(2) 2. $(\forall x)[Gx \rightarrow (\forall z)(Pz \rightarrow \neg Hxz)]$ P
(3) 3. $(\exists x)Gx$ P (beginning existential instantiation)
(1,2,3) 1. $(\forall x)(Fx \rightarrow \neg Px)$
(4) 4. Ga P (beginning existential instantiation)
(1,2,4) 2. $(\forall x)(Fx \rightarrow \neg Px)$ for 1, by existential instantiation
(1,2,4) 3. $(Fb \rightarrow \neg Pb)$ for 2 by UG
(1) 5. $[Ga \rightarrow (\forall y)(Fy \rightarrow Hay)]$ (I16), 1
(2) 6. $[Ga \rightarrow (\forall z)(Pz \rightarrow \neg Haz)]$ (I16), 2
(1,4) 7. $(\forall y)(Fy \rightarrow Hay)$ (I9), 4, 5
(2,4) 8. $(\forall z)(Pz \rightarrow \neg Haz)$ (I9), 4, 6
(1,4) 9. $(Fb \rightarrow Hab)$ (I16), 7
(2,4) 10. $(\neg Pb \rightarrow \neg Hab)$ (I16), 8
(2,4) 11. $(\neg Pb \rightarrow \neg Hab)$ DN, 10
(2,4) 12. $(Hab \rightarrow \neg Pb)$ (E16), 11
(1,2,4) 13. $(Fb \rightarrow \neg Pb)$ (I13), 9, 12 this discharges interest 3
(1,2,4) 14. $(\forall x)(Fx \rightarrow \neg Px)$ UG, 13 this discharges interest 2
(1,2) 15. $[Ga \rightarrow (\forall x)(Fx \rightarrow \neg Px)]$ C, 4, 14
(1,2) 16. $(\forall x)[Gx \rightarrow (\forall x)(Fx \rightarrow \neg Px)]$ UG, 15
17. \[(\exists x)Gx \to (\forall x)Fx \to \neg Px)\] (E24), 16

18. \[(\forall x)(Fx \to \neg Px)\] (I9), 3, 17 (completing existential instantiation) This discharges interest 1

In this derivation, conclusions 1 – 10 and interests 1 – 3 involve eliminating the quantifiers from the premises. Steps 11 – 13 involve getting the desired conclusion without its quantifier. Steps 14 – 18 are concerned with putting the quantifier on by UG, and completing the existential instantiation used to eliminate the quantifier from the third premise.

It should be emphasized that UG and I16 and I17 can only be used with respect to an entire line—never just part of a line of a derivation. If for example, we had a derivation in which the closed formula \[Fa \to (\exists y)Gy\] appeared, we could not obtain \[(\forall x)Fx \to (\exists y)Gy\] by UG, nor could we obtain \[(\exists x)Fx \to (\exists y)Gy\] by I17. Similarly, if \[(\forall x)Fx \to (\exists y)Gy\] appeared on a line of derivation, we could not use I16 to obtain \[Fa \to (\exists y)Gy\]. These restrictions on I16 and I17 are restrictions on Rule I in general. Remember that we can never use Rule I on just part of a line. And a similar restriction applies to Rule UG.

Clearly it will generally be much easier to show that a closed formula is valid by constructing a derivation of it than by giving some sort of general argument involving truth under all interpretations. But, just as in the case of derivations in the propositional calculus, the construction of derivations is not a purely mechanical matter. It will often require considerable ingenuity to find the appropriate derivation. Constructing derivations is not materially different from proving theorems in mathematics. In the light of this, one might desire some purely mechanical means, analogous to truth tables, for evaluating the validity or invalidity of closed formulas. Unfortunately, it can be proven that there is no such mechanical technique, so we will have to be content with using our ingenuity and constructing derivations.

### 3. A Shortcut Rule for Existential Instantiation

When using the strategy of existential instantiation, the last four steps are always the same. We employ rule C, UG, E24, and I9. It becomes tedious to mechanically write out these four steps each time we use the strategy, so it is convenient to introduce a shortcut rule (analogous to DN and R) that allows us to compress those four steps into a single line. For example, in the preceding derivation, this will allow us to move directly from line 14 to line 18, just rewriting the premise numbers:

**Rule EI: Existential Instantiation** If a derivation contains a formula of the form \[(\exists x)A\] on some line \(i\) and on another line \(j\) it contains a premise of the form \(Sb(c/x)A\) where \(c\) does not occur in \(A\), then if a formula \(B\) occurs on any later line and (1) \(c\) does not occur in \(B\), (2) \(j\) is one of the premise numbers of \(B\), and (3) \(c\) does not occur in any of the premises
of \( B \) except for \( j \), then we can write \( B \) on any later line of the derivation, replacing the premise number \( j \) by the premise numbers of line \( i \).

The condition that \( c \) not occur in \( B \) or any of its premises other than \( \text{Sb}(c/x)A \) is required for us to be able to use \( UG \) in inferring \( (\forall x)(A \rightarrow B) \) from \( [\text{Sb}(c/x)A \rightarrow B] \).

Using \( EI \), we can shorten the preceding derivation as follows:

Using \( EI \), we can shorten the preceding derivation as follows:

For each of the following, construct an annotated derivation of the closed formula below the line from the closed formulas above the line. Keep the strategies in mind.

1. \((\forall x)(P_x \rightarrow \neg P_x)\)
   \(\neg P_a\)

2. \((\forall x)[F_x \rightarrow (H_x \rightarrow G_x)]\)
   \((\forall x)F_x\)
   \([[(H_a \rightarrow G_a) \& \neg(-G_b \& H_b)]\)
3. \((\forall x)[P x \leftrightarrow (H x \& \neg P x)]\) (hint: try reductio)  
\[(\forall x)\neg H x\]

4. \([\exists x]F x \to (\forall y)G y\)  
\[(\forall x)(\forall y)(F x \to G y)\]

5. \((\forall x)(F x \to G x)\)  
\[\neg(\forall x)G x\]  
\[(\exists y)\neg F y\]

6. \((\forall x)[F x \to (G x \to H x)]\)  
\[[(\forall x)(F x \to G x) \to (\forall x)(F x \to H x)]\]

7. \((\forall x)(\exists y)R_{xy}\)  
\[(\forall x)(\forall y)(R_{xy} \to R_{yx})\]  
\[(\forall x)(\forall y)(\forall z)(R_{xy} \& R_{yz} \to R_{xz})\]  
\[(\forall x)R_{xx}\]

B. Show that each of the following closed formulas is valid by constructing an annotated derivation of it; that is, a derivation whose last line has no premise numbers:

1. \([\forall x]F x \to (\exists x)F x\] (hint: use reductio ad absurdum)
2. \((\exists x)(F x \to (\forall y)F y)\)
3. \([[(\forall x)(\forall y)(R_{xy} \to \neg R_{yx}) \to (\exists x)R_{xx}]]\)
4. \[\neg(\exists x)(\forall y)(R_{xy} \to \neg R_{yy})\]
5. \[\neg(\forall x)[(F x \lor \neg F x) \to \neg(F x \lor \neg F x)]\]
6. \[\{[\exists x](F x \lor G x) \leftrightarrow [\exists x]F x \lor [\exists x]G x]\}
7. \[\{[\forall x](F x \land G x) \leftrightarrow ([\forall x]F x \land [\forall x]G x]\}
8. \[\{[\forall x](F x \to G x) \to ([\forall x]F x \to [\forall x]G x]\}
9. \[\{[P \to (\exists x)F x] \leftrightarrow (\exists x)(P \to F x)\]
10. \[\{([\forall x]F x \to P) \leftrightarrow (\exists x)(F x \to P)\]

C. Symbolize each of the following arguments in the predicate calculus, and construct an annotated derivation of its conclusion from its premises.

1. Every teacher has some good students. Some students are also teachers. Thus some students have some good students. [T: “(1) teaches
2. Caesar was warned (by someone) to beware the ides of March. If he had been warned by someone reliable, he would have been careful, but he was not careful. Therefore, he was warned by someone unreliable. 
[W: “(1) warned (2) to beware the ides of March”; P: “is a person”; C: “is careful”; R: “is reliable”; c: Caesar]

3. If someone were to steal the royal crown, he would not be able to sell it. If a person stole the royal crown and could not sell it, he would melt it down. If someone melts down the royal crown, he will discover that it is hollow, and know that the prince is a crook. Therefore, if someone steals the royal crown, then someone will know that the prince is a crook. 
[P: “is a person”; S: “(1) steals (2)”]; c: the royal crown; A: “(1) is able to sell (2)”]; M: “(1) melts down (2)”]; D: “(1) discovers (2) is hollow”; K: “(1) knows that (2) is a crook”; p: the prince]

4. Given any positive integer, there is a greater positive integer. Therefore, given any positive integer, there are two positive integers such that the second is greater than the first, and the first is greater than the given integer. 
[P: “is a positive integer”; L: “(1) is larger than (2)”]

5. Some people in England are older than anybody in Scotland. But some people in Siberia are older than anybody in England. If one person is older than a second person, and the second is older than a third, then the first person is older than the third. Therefore, some people in Siberia are older than anybody in Scotland. 
[E: “is a person in England”; S: “is a person in Scotland”; O: “(1) is older than (2)”]; I: “is a person in Siberia”]

D. Try your hand at the following derivations. These are famously difficult derivations taken from the literature on automated theorem proving.1 Some of these are hard enough that even professional logicians and mathematicians may require several hours to construct a derivation.

1. Pelletier’s problem 26

\[(\exists x)Px \leftrightarrow (\exists y)Qy\]
\[(\forall x)(\forall y)((Px \& Qy) \rightarrow (Rx \leftrightarrow Sy))\]
\[[(\forall x)(Px \rightarrow Rx) \leftrightarrow (\forall y)(Qy \rightarrow Sy)]\]

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1 Most of these are from the list compiled by Jeff Pelletier, “Seventy-five problems for testing automatic theorem provers”, Journal of Automated Reasoning 2 (1986), 191-216.
2. Pelletier’s problem 29

\[ (\exists x)Fx \land (\exists y)Gx \]
\[ \equiv (\forall x)(Fx \rightarrow Hx) \land (\forall y)(Gy \rightarrow Jy) \quad \leftrightarrow \quad (\forall z)(\forall w)[(Fz \land Gw) \rightarrow (Hz \land Jw)] \]

3. Pelletier’s problem 33

- no premises -
\[ (\forall x)[(B \land (Fx \rightarrow Pb)) \rightarrow Pc] \leftrightarrow (\forall x)[(\neg Pa \lor (Fx \lor Pc)) \land (\neg Pa \lor (\neg Pb \lor Pc))] \]

4. Pelletier’s problem 34 (Andrew’s Challenge)

- no premises -
\[ [(\exists x)(\forall y)(Fx \leftrightarrow Py) \leftrightarrow (\exists z)(Qz \leftrightarrow (\forall w)Qw)] \leftrightarrow [(\exists u)(\forall v)Qu \leftrightarrow Qv] \leftrightarrow (\exists x)[Fx \leftrightarrow (\forall y)Py] \]

5. Pelletier’s problem 37

\[ (\forall z)(\exists w)(\forall x)(\exists y)[(Pxy \rightarrow Pyw) \land (Pyw \rightarrow (\exists u)Quw)] \]
\[ (\forall x)(\forall z)[\neg Pxz \rightarrow (\exists v)Qvz] \]
\[ (\exists y)(\exists z)(Qyz \rightarrow (\forall x)Rxx) \]
\[ (\exists x)(\forall y)Rxy \]

6. Pelletier’s problem 42

A set is “circular” if it is a member of another set which in turn is a member of the original. Pelletier observes that, intuitively, all sets are non-circular. The problem is to show that there is no set of noncircular sets.

- no premises -
\[ \neg (\exists y)(\forall x)[Fx \leftrightarrow (\exists z)(Fx \land Fxz)] \]

7. Pelletier’s problem 43

Define set equality (Q) as having exactly the same members. Prove set equality is symmetric.

\[ (\forall x)(\forall y)[Qxy \leftrightarrow (\forall z)(Fzx \leftrightarrow Fzy)] \]
\[ (\forall x)(\forall y)[Qxy \leftrightarrow Qyx] \]
8. Pelletier’s problem 45

\((\forall x)((Fx \& (\forall y)((Gy \& Hxy) \rightarrow Jxy)) \rightarrow (\forall y)((Gy \& Hxy) \rightarrow Ky))\)
\(~(\exists y)(Ly \& Ky)\)
\((\exists x)((Fx \& (\forall y)((Hxy \rightarrow Ly)) \& (\forall y)((Gy \& Hxy) \rightarrow Jxy))\)
\((\exists x)(Fx \& \neg(\exists y)(Gy \& Hxy))\)

9. Pelletier’s problem 46

\((\forall x)((Fx \& (\forall y)((Fy \& Hyx) \rightarrow Gy)) \rightarrow Gx)\)
\([\exists x](Fx \& \neg Gx) \rightarrow (\exists x)((Fx \& \neg Gx) \& (\forall y)((Fy \& \neg Gy) \rightarrow Jxy)]\)
\((\forall x)(\forall y)((Fx \& Hxy) \rightarrow \neg Jyx)\)
\((\forall x)(\forall y)((Fx \rightarrow Gy) \rightarrow Gx)\)

10. Pelletier’s problem 47. This is known as the Schubert Steamroller problem. This is a slightly whimsical symbolization of the following:

Wolves, foxes, birds, caterpillars, and snails are animals, and there are some of each of them. Also, there are some grains, and grains are plants. Every animal either likes to eat all plants or all animals much smaller than itself that like to eat some plants. Caterpillars and snails are much smaller than birds, which are much smaller than foxes, which in turn are much smaller than wolves. Wolves do not like to eat foxes or grains, while birds like to eat caterpillars but not snails. Caterpillars and snails like to eat some plants. Therefore, there is an animal that likes to eat a grain-eating animal.

\((\forall x)(Wx \rightarrow Ax)\)
\((\forall x)(Fx \rightarrow Ax)\)
\((\forall x)(Bx \rightarrow Ax)\)
\((\forall x)(Cx \rightarrow Ax)\)
\((\forall x)(Sx \rightarrow Ax)\)
\((\exists w)Ww\)
\((\exists f)Ff\)
\((\exists b)Bb\)
\((\exists c)Cc\)
\((\exists s)Ss\)
\((\exists g)Gg\)

11. The “unintuitive problem”. This problem is of interest because when presented to most logicians or mathematicians, their immediate reaction is “That cannot be right”.

\((\forall x)(Ax \rightarrow [(\forall w)(Pw \rightarrow Exw) \rightarrow (\forall y)((Ay \& (Myx \& (\exists z)(Pz \& Eyz))) \rightarrow Exy))]\)
\((\exists x)(\exists y)((Ax \& Ay) \& (\exists z)[Exy \& (Gz \& Eyz)])\)
(∀x)(∀y)(∀z)[(Px·y & Py·z) → Px·z]
(∀x)(∀y)(∀z)[(Qxy & Qyz) → Qxz]
(∀x)(∀y)(Qxy → Qyx)
(∀x)(∀y)(¬Pxy → Qxy)

~Pab

Qcd

12. This is a problem in group theory taken from Chang and Lee, *Symbolic Logic and Mechanical Theorem Proving*, Academic Press, 1973. Pxyz symbolizes “x·y = z”.

(∀x)Pxex
(∀x)Pexx
(∀x)(∀y)(∀z)(∀u)(∀v)(∀w)[[Pxy & (Py·z & Puz·w)] → Px·vw]
(∀x)(∀y)(∀z)(∀u)(∀v)(∀w)[[Pxy & (Py·z & Px·vw)] → Puz·ww]
(∀x)Pxex

Pabc

Pbac