3
The Semantics of the Propositional Calculus

1. Interpretations

Formulas of the propositional calculus express statement forms. In chapter two, we gave informal descriptions of the meanings of the logical symbols of the propositional calculus, and relied upon that for our understanding of the statement forms expressed by formulas. The next step is to give a more precise description of those meanings. The objective is to give a description that is sufficiently precise to allow us to use mathematical tools in studying formal necessity and the validity of argument forms.

Our general approach will be the same as for the implicational calculus. We will begin by defining the notion of an *interpretation*, and then we will give truth rules for the logical symbols. It is important to bear in mind that formulas do not express statements—they express statement forms. As such, they are not true or false. Typically, one and the same formula will be the form of both true statements and false statements. For example, $(P \vee Q)$ is the form of both of the following statements:

Either mice have whiskers or worms have wings.

Either mice have wings or worms have whiskers.

But the first statement is true and the second false.

This means that a truth value can only be associated with a formula by first assigning meanings to its atomic parts. Once we do that, we pick out a particular statement rather than just a statement form. We can say that the statement form is true or false *relative to an assignment of meanings* if, and only if, the statement that results from that assignment of meanings is true or false. In other words, attributing truth values to formulas (relative to interpretations) is just a shorthand way of talking about the truth values of corresponding statements.

The most natural way to define the notion of an interpretation would be to identify it with an assignment of statements to sentential letters. Once we have said what statements the sentential letters stand for, it is determined what statement is expressed by formulas built out of those sentential letters. However, formulas of the propositional calculus have the nice characteristic that we do not have to know their complete meanings to know their truth values. For instance, consider a conjunction (P & Q). If we know that P and Q are both true, then we can conclude that (P & Q) is true, and we can do this without knowing what statements P and Q express. It will turn out that this is true for all formulas of the propositional calculus. For this reason, we will take an interpretation to simply assign truth values to sentential letters rather than assign entire meanings. So:

> An *interpretation of the propositional calculus* is an assignment of truth values (truth or falsity) to the sentential letters.

We can then talk about formulas being true or false *relative to an interpretation*.

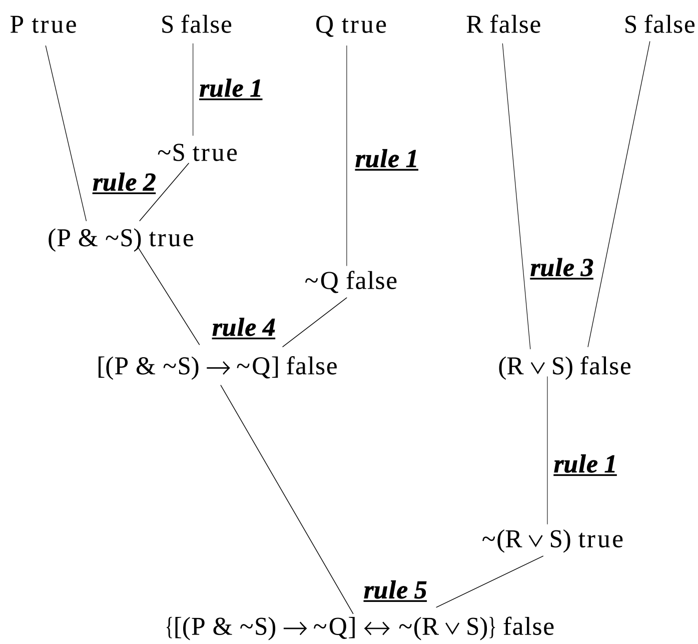
2. Truth Rules

To verify the claim that we can compute the truth value of any formula if we know the truth values of the sentential letters occurring in it, we must give the truth rules that allow us to make that computation. The rules we employ are the following:

> If A is any formula, ~A is true iff A is false.
>
> If A and B are any formulas, (A & B) is true if and only if both A and B are true.
>
> If A and B are any formulas, ($A \vee B$) is true if and only if either A or B (or both) are true.
>
> If A and B are any formulas, ($A \rightarrow B$) is true if and only if either A is false or B is true.
>
> If A and B are any formulas, ($A \leftrightarrow B$) is true if and only A and B have the same truth value, i.e., either both are true or both are false.

The truth rules for all but the conditional are self-explanatory, and the truth rule for the conditional is just the definition of the material conditional, as we discussed in chapter two.

Repeated application of these truth rules allows us to determine the truth or falsity of complex formulas. To illustrate, suppose we are given that P and Q are true, and R and S are false, and we want to determine whether the formula {[(P & ~S) → ~Q] ↔ ~(R ∨ S)} is true. We begin with the smallest parts. S is false, so ~S is true. Then P and ~S are both true, so by Rule 2, (P & ~S) is true. Q is true, so by Rule 1, ~Q is false. Then by Rule 4, as (P & ~S) is true and ~Q is false, the conditional [(P & ~S) → ~Q] is false. Neither R nor S is true, so by Rule 3, (R ∨ S) is false. Then by Rule 1, ~(R ∨ S) is true. But then the two sides of the biconditional {[(P & ~S) → ~Q] ↔ ~(R ∨ S)} have different truth values (the left side is false and the right side is true), and so, by Rule 5, the biconditional is false. We can diagram this procedure as follows:



Given the truth values of the atomic parts of a formula, it becomes a mechanical matter to calculate the truth value of the whole formula using the above truth rules. We begin with the smallest parts of the formula, calculate their truth values, and then work outwards until we get the whole formula. The calculation of the truth value parallels the construction we would use in building the formula from its atomic parts.

3. Truth Tables

The truth rules allow us to compute the truth value of any formula relative to an interpretation. There are infinitely many sentential letters, because we allow the use of numerical subscripts. It follows that there are infinitely many interpretations of the propositional calculus. However, when focusing on a single formula, the only part of an interpretation that is relevant to its truth value is the assignment to the sentential letters that are atomic parts of the formula. There will be only finitely many ways interpretations can assign truth values to any finite set of sentential letters. For example, if we consider the sentential letters P, Q, and R, we can tabulate all the ways of assigning truth values to them as follows:

P	Q	R
T	T	T
T	T	F
T	F	T
T	F	F
F	T	T
F	T	F
F	F	T
F	F	F

Note how we arranged the truth values to ensure that we have an exhaustive list. In the rightmost column (under R), the truth values alternate. In the next column (under Q) they alternate two at a time. In the final column (under P) they alternate four at a time. If there were another column, they would alternate eight at a time, and so on. In general, given a list of *n* sentential letters, there are 2^n combinations of truth values that can be assigned to them. Thus for two senntentential letters there are four combinations of truth values, for three sentential letters there are eight, for four there are sixteen, and so forth.

We can express the truth rules for the logical symbols in tabular form as follows:

P	~P
T	F
F	T

P	Q	(P & Q)
T	T	T
T	F	F
F	T	F
F	F	F

P	Q	$(P \vee Q)$
T	T	T
T	F	T
F	T	T
F	F	F

P	Q	$(P \rightarrow Q)$
T	T	T
T	F	F
F	T	T
F	F	T

P	Q	$(P \leftrightarrow Q)$
T	T	T
T	F	F
F	T	F
F	F	T

In a similar way, we can tabulate the truth values of that formula for all possible combinations of truth values of its atomic parts. Consider the formula $[(P \& {\sim}Q) \leftrightarrow ({\sim}(P \vee Q) \rightarrow Q)]$. The following table can be constructed. At each step the column of truth values being computed is greyed out, and the columns from which the computations is made are marked in a lighter grey:

Step 1

P	Q	$[(P \& {\sim}Q) \leftrightarrow ({\sim}(P \vee Q) \rightarrow Q)]$
T	T	
T	F	
F	T	
F	F	

Step 2 Truth values for ~Q

P	Q	$[(P \&$	${\sim}Q)$	$\leftrightarrow ({\sim}(P \vee Q) \rightarrow Q)]$
T	T		F	
T	F		T	
F	T		F	
F	F		T	

Step 3 Truth values for (P & ~Q).

P	Q	$[(P$	$\&$	${\sim}Q)$	$\leftrightarrow ({\sim}(P \vee Q) \rightarrow Q)]$
T	T		F	F	
T	F		T	T	
F	T		F	F	
F	F		F	T	

Step 4 Truth values for (P ∨ Q)

P	Q	[(P &	~Q)	↔	(~	(P ∨ Q)	→ Q)]
T	T	F	F			T	
T	F	T	T			T	
F	T	F	F			T	
F	F	F	T			F	

Step 5 Truth values for ~(P ∨ Q)

P	Q	[(P &	~Q)	↔	(~	(P ∨ Q)	→ Q)]
T	T	F	F		F	T	
T	F	T	T		F	T	
F	T	F	F		F	T	
F	F	F	T		T	F	

Step 6 Truth values for (~(P ∨ Q) → Q)

P	Q	[(P &	~Q)	↔	(~	(P ∨ Q)	→ Q)]
T	T	F	F		F	T	T
T	F	T	T		F	T	T
F	T	F	F		F	T	T
F	F	F	T		T	F	F

Step 7 Truth values for [(P & ~Q) ↔ (~(P ∨ Q) → Q)]

P	Q	[(P &	~Q)	↔	(~	(P ∨ Q)	→ Q)]
T	T	F	F	***F***	F	T	T
T	F	T	T	***T***	F	T	T
F	T	F	F	***F***	F	T	T
F	F	F	T	***T***	T	F	F

This table is called a *truth table*. On the left-hand side of a truth table we list all possible combinations of truth values for the atomic parts of the formula. On the right-hand side are columns corresponding to each occurrence of a logical symbol in the formula, and in that column we list the truth value of the part of the formula that is constructed using that connective. We enclose the column of truth values that are the truth values of the entire formula in a box and print it in bold italic type.

The truth table for a formula gives us a record of its truth value under all possible interpretations. The only thing difficult about constructing truth tables is keeping track of which columns to use in computing the truth values for each new column. It is easy to become confused about that. The best way to avoid confusion is to keep in mind how the formula is constructed. For example, in step 7 we are computing the truth value of a biconditional. Ask yourself what the left and right sides of the biconditional are, and which columns record their truth values. Those are then the columns to which you appeal in computing the truths values of the biconditional. You may find it useful to connect matching parentheses with lines, as was done on page 33.

Exercises

Construct truth tables for the following formulas:

1. $\sim(P \rightarrow \sim\sim P)$
2. $[(P \rightarrow Q) \vee \sim(P \ \& \sim Q)]$
3. $[(P \leftrightarrow Q) \leftrightarrow \sim(P \leftrightarrow \sim Q)]$
4. $[((P \ \& \ Q) \rightarrow R) \ \& \sim((R \vee P) \leftrightarrow \sim(P \rightarrow Q))]$
5. $[\ \sim(P \rightarrow Q) \rightarrow \sim([(R \vee P) \ \& \sim(Q \rightarrow \sim P)] \leftrightarrow \sim[(R \ \& \sim Q) \rightarrow (P \ \& \sim R)])]$
6. $[\ \sim(P \ \& \ Q) \leftrightarrow (\sim P \ \& \sim Q)]$

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4. Tautologies

Our main interest is in the formal necessity of statement forms and the formal soundness of argument forms. Using truth tables we can give very simple characterizations of these concepts for statement forms and argument forms that can be symbolized in the propositional calculus. Let us begin by discussing formal necessity.

Consider the logically necessary statement “Either it is snowing or it is not snowing”. This statement has the form $(P \vee \sim P)$. Any statement that has this form will be necessary. Why is this so? Because regardless of what we let P mean, either P or ~P will be true, and so (P ∨ ~P) will be true. To make this clearer, consider the truth table for (P ∨ ~P):

P	(P ∨ ~P)	
T	***T***	F
F	***T***	T

Notice that (P ∨ ~P) comes out true on every line of this truth table. This is enough to make it formally necessary, because regardless of how we interpret P, it will have one of the truth values listed in the truth table, and then (P ∨ ~P) will be true. Thus we can see by examining the truth table that any statement having the form (P ∨ ~P) will be true under all circumstances, regardless of whether P is true or false. This means that the statement is necessary. So any statement of the form (P ∨ ~P) is necessary, and hence this statement form is formally necessary.

It was remarked in chapter one that a formula true under every interpretation is said to be *valid*. In the propositional calculus, a formula is valid if, and only if, it is true on every line of its truth table. Such formulas are called *tautologies*. (P ∨ ~P) is therefore a tautology. We write ⊦ *A* to indicate that a formula *A* is a tautology. For example, we can write ⊦ (P ∨ ~P).

The significance of tautologies is that they are the formally necessary statement forms of the propositional calculus. To see this, suppose we have some formula of that propositional calculus that is a tautology, such as {[P → (Q → R)] ↔ [(P & Q) → R]}. Consider any statement that has this form. Then, even without knowing whether P, Q, and R are true, we can verify that the statement itself is true just by seeing that the formula comes out true on every line of its truth table. This is because, regardless of what statements are symbolized by P, Q, and R, they will have *some* truth values, and those truth values will correspond to a line of the truth table. The statement symbolized by [P → (Q → R)] ↔ [(P & Q) → R]} will then be true if and only if the formula is true on that line of its truth table. But the formula is true on every line of its truth table, so it follows that the statement will be true regardless of how the world might be (regardless of the truth values of P, Q, and R). In other words, any statement of this form will be necessary, and the statement form will be formally necessary. Therefore, we can conclude:

> If a formula is a tautology then the statement form it symbolizes is formally necessary.

The converse of this principle is also true:

> If a formula is a not tautology then the statement form it symbolizes is not formally necessary.

This is because for every line of the truth table there will be *some* statement having the form symbolized by the formula and having the truth value the formula has on that line. To illustrate, consider the formula [P → (P & Q)]. This is not a tautology. It will be false when P is true but Q is false. But now pick any statements having the truth values that, when assigned to P and Q, make [P → (P & Q)] false. For instance, we might let P stand for "2+2 = 4" and Q stand for "2+3 = 6". The statement

[2+2 = 4 → (2+2 = 4 & 2+3 = 6)]

is then be true if and only if [P → (P & Q)] is true on that line of its truth table where P is true and Q is false. Hence the statement is false. It follows that not all statements of the form [P → (P & Q)] are logically necessary (in particular, this one isn't), and so [P → (P & Q)] does not symbolize a formally necessary statement form.

We can conclude in general then that:

> A formula of the propositional calculus is a tautology if, and only if, it is the form of a formally necessary statement form.

Thus truth tables and tautologies give us a way of talking about the formal necessity of any statement whose form can be expressed in the propositional calculus.

5. Metatheorems

Logical necessity was explained using metaphors and by giving examples, but not by giving a precise definition. That has the consequence that although we can reason about the properties of logical necessity and the attendant concepts of formal necessity and the validity of argument forms, we cannot literally *prove theorems* about these concepts in the sense that one proves theorems in mathematics. In this respect, they contrast sharply with the concept of a tautology. The concept of a tautology is a mathematically precise concept. We could, for example, write a computer program that mechanically checks formulas to determine whether they are tautologies. It would do this by constructing a truth table.

Because tautologicity is a mathematically precise concept, we can literally prove mathematical theorems about it. For example, one important fact about tautologies is:

> The conjunction of two tautologies is itself a tautology.

That is, if we begin with two formulas, A and B, and $\vdash A$ and $\vdash B$, then $\vdash (A \;\&\; B)$. This principle is known as *adjunctivity*. To establish this, suppose that $\vdash A$ and $\vdash B$. Then any assignment of truth values to the atomic parts of A and B will make them both true. But then any such assignment of truth values will also make the conjunction (A & B) true, and so it is also a tautology.

The preceding reasoning is simple, but the result is quite literally a mathematical theorem about tautologicity. Theorems about a logical theory are called *metatheorems*. The investigation of a logical theory proceeds largely by proving metatheorems. It is the metatheorems that enable us to determine the properties of the theory and investigate how, for example, to construct arguments using formulas of the theory. Metatheorems need not be very deep, and their proofs are often easy. This is illustrated by the proof of adjunctivity. But even simple metatheorems can be important.

We have two classes of concepts pertaining to the propositional calculus. We started off with the philosophical concepts of logical necessity, validity, formal necessity, and the validity of argument forms. Then we introduced the mathematical concept of tautologicity, and we will use it to define other mathematical concepts. The philosophical concepts represent our initial interest in the propositional calculus, but the mathematical concepts provide a tool for studying the philosophical concepts. This is made possible by the argument given in the previous section to the effect that for formulas of the propositional calculus, tautologicity and formal necessity coincide. Thus we can study the philosophically interesting concept of formal necessity *by* studying the mathematical concept of tautologicity, and we can use all of the tools of mathematics in studying the latter. The mathematics required need not be very difficult, but what it achieves is rigorous proofs of precise results.

By proving metatheorems, we become clearer on the logical structure of the propositional calculus, and this will eventually enable us to prove important results about how to reason using formulas of the propositional calculus. The metatheorems comprise what is called *the metatheory* of the propositional calculus. The metatheory is just the collection of all our metatheorems.

To advance our study of the propositional calculus, we can define two concepts that are closely related to tautologicity. We say that a formula is *truth-functionally consistent* if, and only if, it is true on at least one line of its truth table. A formula is *truth-functionally inconsistent* if, and only if, it is not truth-functionally consistent—that is, if and only if it is false on every line of its truth table. Because these concepts are defined in terms of tautologicity, they are also mathematically precise and we can prove metatheorems about them. For instance, here is a simple metatheorem about truth-functional inconsistency:

> ***Metatheorem***: A formula is truth-functionally inconsistent if, and only if, its negation is a tautology.

Proof: A truth-functionally inconsistent formula is one that is false on every line of its truth table. A formula is false on every line of its truth table if, and only if, its negation is true on every line of its truth table. But the negation is true on every line of its truth table if, and only if, it is a tautology. ■

Note that "■" marks the end of the proof.

We can prove many similar metatheorems by thinking about truth tables and assignments of truth values to the atomic parts of formulas. Usually, such a proof begins by "expanding the definitions" of the terms employed in the metatheorem. For example, the above proof began by expanding the definitions of truth-functional inconsistency and tautologicity. Then we reason about the conditions that are stated in the expanded form of the principle being proven. The student should keep this general strategy in mind in doing the following exercises.

Exercises

A. Prove the following metatheorems by giving arguments analogous to those given in the text.

1. Every tautology is truth-functionally consistent.
2. If one formula is a tautology, and a second formula is truth-functionally consistent, then their conjunction is truth-functionally consistent.

B. To prove that something holds in general one must give a general argument. However, in the following exercises, you are asked to prove that something is possible. This can be done by giving a single example.

1. It is possible for two formulas to be each truth-functionally consistent while their conjunction is truth-functionally inconsistent.
2. It is possible for both a formula and its negation to be truth-functionally consistent.

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6. Tautological Implication

The concept of tautologicity provides the vehicle for a mathematical study of formal necessity in the propositional calculus. We can define an analogous concept that allows us to study the validity of argument forms expressed in the propositional calculus. We saw in chapter one that an argument form is valid if, and only if, its corresponding conditional is formally necessary. In the propositional calculus, formal necessity coincides with tautologicity, so it follows that:

> An argument form expressed in the propositional calculus is valid if, and only if, its corresponding condition is a tautology.

This is captured by the concept of *tautological implication*. We define:

> A set of formulas $A_1,\ldots,A_n$ tautologically implies a formula B if, and only if, $\vdash [(A_1 \& \ldots \& A_n) \rightarrow B]$.

We abbreviate "$A_1,\ldots,A_n$ tautologically implies B" as "$A_1,\ldots,A_n \vdash B$".

The significance of tautological implication is given by the following principle:

> An argument form expressed in the propositional calculus is valid if, and only if, its premises tautologically imply its conclusion.

Tautological implication consists of the tautologicity of the corresponding conditional, and that can be checked by constructing a truth table. So given an argument form expressed in the propositional calculus, we can assess its validity by constructing a truth table and determining whether its corresponding conditional is a tautology.

A conditional is a tautology just in case its conclusion is true on every line of its truth table on which its antecedent is true. Thus an argument form expressed in the propositional calculus is valid (its premises tautologically imply its conclusion) if, and only if, every line of its truth table on which its premises are true is also a line on which its conclusion is true. Hence truth tables become a valuable tool for investigating the validity of argument forms. It has to be emphasized, however, that the scope of this tool is limited to argument forms that can be expressed in the propositional calculus. There are many valid argument forms that cannot be expressed in the propositional calculus. For example:

All A are B.
All B are C.

Therefore, all A are C

is a valid argument form, but it cannot be expressed in the propositional calculus. To deal with arguments like this we need the greater expressive power of the predicate calculus, which will be investigated in part II of the book. These arguments cannot be assessed using truth tables.

It will be useful to have a stock of tautological implications to which we can appeal when we begin investigating how to reason and construct arguments within the propositional calculus. The following tautological implications are important and are easily verified using truth tables:

$$\left.\begin{array}{ll}\text{I1.} & (A \,\&\, B) \vdash A \\ \text{I2.} & (A \,\&\, B) \vdash B\end{array}\right\} \textit{simplification}$$

$$\left.\begin{array}{ll}\text{I3.} & A \vdash (A \vee B) \\ \text{I4.} & B \vdash (A \vee B)\end{array}\right\} \textit{addition}$$

I5. $\sim A \vdash (A \rightarrow B)$

I6. $B \vdash (A \rightarrow B)$

I7. $\sim(A \rightarrow B) \vdash A$

I8. $\sim(A \to B) \vdash \sim B$

I9. $A, (A \to B) \vdash B$ *modus ponens*

I10. $\sim B, (A \to B) \vdash \sim A$ *modus tollens*

$$\left.\begin{array}{l}\text{I11. } \sim A, (A \lor B) \vdash B \\ \text{I12. } \sim B, (A \lor B) \vdash A\end{array}\right\} \textit{disjunctive syllogism}$$

I13. $(A \to B), (B \to C) \vdash (A \to C)$ *hypothetical syllogism*

I14. $A, B \vdash (A \,\&\, B)$ *adjunction*

I15. $(A \lor B), (A \to C), (B \to C) \vdash C$ *dilemma*

Let us see why these implications hold:

I1 and I2 both hold because a conjunction is true if and only if both conjuncts are true.

I3 and I4 hold because a disjunction is true if either disjunct is true.

I5 holds because a conditional is true if its antecedent is false.

I6 holds because a conditional is true if its consequent is true.

I7 and I8 hold because a conditional is only false when its antecedent is true and its consequent false.

I9: Suppose A is true and $(A \to B)$ is true. Then we have a true conditional having a true antecedent. But then, in order for the conditional to be true, the consequent must also be true. Thus B must be true.

I10: Exercise for the student.

I11: Suppose $\sim A$ and $(A \lor B)$ are both true. In order for the disjunction to be true, at least one of its disjuncts must be true. But the first disjunct is false, because $\sim A$ is true. Thus the second disjunct, B, must be true. I12 is analogous.

I13: Exercise for the student.

I14: If both A and B are true, then both conjuncts of $(A \,\&\, B)$ are true, so the conjunction is true. Note that although this principle is called *adjunction*, it is a different principle from the adjunctivity of tautologicity. This principle is the *adjunctivity of tautological implication*.

I15: Suppose $(A \lor B)$, $(A \to C)$, and $(B \to C)$ are all true. Since the disjunction $(A \lor B)$ is true, either A is true or B is true. But if A is true then C is true (by the first conditional), and if B is true then C is true (by the second conditional). Thus if either A or B is true, then C is true. But we know that either A is true or B is true, so C is true.

Exercises

Prove that I10 and I13 hold

7. Tautological Equivalence

On several occasions it has been remarked that one formula is "equivalent" to another. For instance, it was remarked that "Neither P nor Q" could be symbolized as either (~P & ~Q) or ~(P ∨ Q), and that the two formulas were equivalent. Now we are in a position to make this concept of equivalence precise. The sense in which two formulas of the propositional calculus can be equivalent is that they are true under the same circumstances. More precisely, they are true relative to the same interpretations. This concept of equivalence is called *tautological equivalence*:

> Two formulas of the propositional calculus are *tautologically equivalent* if, and only if, they are true relative to the same interpretations.

Thus, for example, (~P & ~Q) and ~(P ∨ Q) are true relative to the same interpretations. This can be verified by constructing a truth table:

P	Q	(~P & ~Q)			~(P ∨ Q)	
T	T	F	***F***	F	***F***	T
T	F	F	***F***	T	***F***	T
F	T	T	***F***	F	***F***	T
F	F	T	***T***	T	***T***	F

If two formulas have the same truth values, then their biconditional is true. E.g., we could rewrite the preceding truth table as follows:

P	Q	[(~P & ~Q)			↔	~(P ∨ Q)]	
T	T	F	F	F	***T***	F	T
T	F	F	F	T	***T***	F	T
F	T	T	F	F	***T***	F	T
F	F	T	T	T	***T***	T	F

Thus we could define tautological equivalence just as well as follows:

> Two formulas of the propositional calculus, A and B, are tautologically equivalent if, and only if, $(A \leftrightarrow B)$ is a tautology.

We will often abbreviate "tautologically equivalent" as "eq." The following tautological equivalences can all be verified very easily using truth tables:

E1.	$\sim\sim A$ eq. A	*double negation*
E2.	$\sim(A \vee B)$ eq. $(\sim A \ \& \sim B)$	*De Morgan's Laws*
E3.	$\sim(A \ \& \ B)$ eq. $(\sim A \vee \sim B)$	
E4.	$(A \rightarrow B)$ eq. $(\sim A \vee B)$	
E5.	$\sim(A \rightarrow B)$ eq. $(A \ \& \sim B)$	
E6.	$\sim(A \leftrightarrow B)$ eq. $(A \leftrightarrow \sim B)$	
E7.	$(A \leftrightarrow B)$ eq. $[(A \rightarrow B) \ \& \ (B \rightarrow A)]$	
E8.	$(A \leftrightarrow B)$ eq. $[(A \ \& \ B) \vee (\sim A \ \& \sim B)]$	
E9.	$(A \ \& \ (B \ \& \ C))$ eq. $((A \ \& \ B) \ \& \ C)$	*associative laws*
E10.	$(A \vee (B \vee C))$ eq. $((A \vee B) \vee C)$	
E11.	$(A \ \& \ B)$ eq. $(B \ \& \ A)$	*commutative laws*
E12.	$(A \vee B)$ eq. $(B \vee A)$	
E13.	$[A \ \& \ (B \vee C)]$ eq. $[(A \ \& \ B) \vee (A \ \& \ C)]$	*distributive laws*
E14.	$[A \vee (B \ \& \ C)]$ eq. $[(A \vee B) \ \& \ (A \vee C)]$	
E15.	$[A \rightarrow (B \rightarrow C)]$ eq. $[(A \ \& \ B) \rightarrow C]$	*exportation-importation*
E16.	$(A \rightarrow B)$ eq. $(\sim B \rightarrow \sim A)$	*contraposition*
E17.	$(A \vee A)$ eq. A	*idempotency laws*
E18.	$(A \ \& \ A)$ eq. A	
E19.	$(A \vee B)$ eq. $(\sim A \rightarrow B)$	
E20.	$(A \vee B)$ eq. $(\sim B \rightarrow A)$	

Let us examine each of these equivalences separately, and see why it is true:

E1. $\sim\sim A$ is true if, and only if, $\sim A$ is not true, and $\sim A$ is not true if, and only if, it is not the case that A is not true; that is, if and only if A is true.

E2. $\sim(A \vee B)$ is true if, and only if, $(A \vee B)$ is false. But $(A \vee B)$ is false if, and only if, both disjuncts are false; that is, if and only if both A and B are false. But this is the same as saying that $\sim A$ and $\sim B$ are both true; that is, that $(\sim A \;\&\; \sim B)$ is true.

E3. $\sim(A \;\&\; B)$ is true if, and only if, $(A \;\&\; B)$ is false. But $(A \;\&\; B)$ is false if, and only if, at least one conjunct is false; that is, if and only if either A is false or B is false. But this is the same as saying that either $\sim A$ is true or $\sim B$ is true; that is, that $(\sim A \vee \sim B)$ is true.

E4. We saw in Section 4 that $(A \rightarrow B)$ is true if, and only if, $(\sim A \vee B)$ is true.

E5. $\sim(A \rightarrow B)$ is true if, and only if, $(A \rightarrow B)$ is false. $(A \rightarrow B)$ is false if, and only if, the antecedent is true and the consequent is false, that is, A and $\sim B$ are true. Thus $(A \rightarrow B)$ is false if, and only if, $(A \;\&\; \sim B)$ is true. Thus, $\sim(A \rightarrow B)$ is true if, and only if, $(A \;\&\; \sim B)$ is true.

E6. $\sim(A \leftrightarrow B)$ is true if, and only if, A and B have opposite truth values; that is, if and only if A and $\sim B$ have the same truth values; that is, if and only if $(A \leftrightarrow \sim B)$ is true.

E7. $(A \leftrightarrow B)$ is true if, and only if, A and B have the same truth value. Thus if A is true then B must be true, and if B is true then A must be true. So if $(A \leftrightarrow B)$ is true, then

$$[(A \rightarrow B) \;\&\; (B \rightarrow A)]$$

must be true. Conversely, if $[(A \rightarrow B) \;\&\; (B \rightarrow A)]$ is true, then if A is true, so must B be true, and if A is false, so must B be false. Thus $(A \leftrightarrow B)$ is true. Therefore, $(A \leftrightarrow B)$ is true if, and only if, $[(A \rightarrow B) \;\&\; (B \rightarrow A)]$ is true.

E8. Exercise for the student.

E9 is obvious.

E10 is obvious.

E11 is obvious.

E12 is obvious.

E13. To say that $[A \;\&\; (B \vee C)]$ is true is to say that both A and $(B \vee C)$ are true. Thus A must be true, and either B or C must be true. So either $(A \;\&\; B)$ will be true, or $(A \;\&\; C)$ will be true. So if $[A \;\&\; (B \vee C)]$ is true, then $[(A \;\&\; B) \vee (A \;\&\; C)]$ is true.

Conversely, if $[(A \;\&\; B) \vee (A \;\&\; C)]$ is true, then either $(A \;\&\; B)$ or $(A \;\&\; C)$ must be true. In either case, A must be true. In addition, in order to have one of $(A \;\&\; B)$ or $(A \;\&\; C)$ true, we must have either B true or C true. So $(B \vee C)$ must be true. Thus both A and $(B \vee C)$ must be true, so $[A \;\&\; (B \vee C)]$ must be true. Thus $[A \;\&\; (B \vee C)]$ is true if, and only if, $[(A \;\&\; B) \vee (A \;\&\; C)]$ is true.

E14. Verify this with a truth table.

E15. Exercise for the student.

E16. If $(A \rightarrow B)$ is true, then if B is false, A must also be false, because if A were true, then the conditional would have a true antecedent and false consequent, which would make it false. Thus if $\sim B$ is true, then $\sim A$ is true; that is, $(\sim B \rightarrow \sim A)$ is true.

Conversely, if $(\sim B \rightarrow \sim A)$ is true, then by the same argument, $(\sim\sim A \rightarrow \sim\sim B)$ must be true. But $\sim\sim A$ is equivalent to A, and $\sim\sim B$ is equivalent to B, so $(A \rightarrow B)$ must be true.

E17 is obvious.

E18 is obvious.

E19. By E1, $(A \vee B)$ is equivalent to $(\sim\sim A \vee B)$, and by E4 this is equivalent to $(\sim A \rightarrow B)$.

E20 is analogous to E19.

Exercises

A. Given arguments showing that E8 and E15 hold.

B. Prove the following metatheorem:

> Two formulas of the propositional calculus are tautologically equivalent if, and only if, they tautologically imply each other.

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8. Some Metatheorems

Tautologicity, tautological implication, and tautological equivalence are the main semantical concepts of the propositional calculus. They have each received characterizations in terms of truth tables. In principle, any claim of tautologicity, tautological implication, or tautological equivalence can be verified by constructing a truth table. However, in practice that is often too difficult. For example, a truth table for an implication involving 8 atomic parts would require a truth table with 256 lines. It would be impractical to try to construct such a truth table. An implication involving 16 atomic parts would require a truth table with 65536 lines. A computer could construct such a truth table, but a human being would probably get it wrong. An implication involving 300 atomic parts would require a truth table with approximately 10^{90} lines. Not even a computer could construct that truth table. It has been estimated that there are 10^{78} elementary particles in the universe, so the truth table would have more lines than there are particles in the universe. And yet, it is not out of the question that we could show such an implication to hold. For example, it might have the form

$(P_1 \& ... \& P_{300}) \vdash P_1$.

We can give a very simple argument to show that this implication holds, without having to construct a truth table.

The preceding example shows that it is often better to reason about tautologicity and tautological implication rather than constructing truth tables. Reasoning about these concepts consists of proving metatheorems. So let us turn to some metatheorems that will subsequently prove useful. Let us begin with a simple one:

> ***Metatheorem***: If a formula is either tautologically equivalent to, or tautologically implied by a tautology, then it is itself a tautology.

Proof: This holds because, if if a formula is tautologically implied by a tautology, then the formula must be true on every line of its truth table on which the tautology is true. Since the tautology is true on every line of the truth table, the formula itself must also be true on every line of its truth table; thus it is a tautology. ■

Next note that *tautological implication is transitive*:

> ***Metatheorem***: If one formula (or set of formulas) tautologically implies a second formula, and the second formula tautologically implies a third, then the first formula (or set of formulas) tautologically implies the third.

Proof: Suppose we have a set of formulas $A_1,\ldots,A_n$ and B and C, and suppose $A_1,\ldots,A_n \vdash B$ and $B \vdash C$. Let us show that it follows from this that $A_1,\ldots,A_n \vdash C$. By the definition of tautological implication, as $A_1,\ldots,A_n \vdash B$ and $B \vdash C$, C is true on every line of the truth table on which B is true, and B is true on every line on which $A_1,\ldots,A_n$ are true. Hence C is true on every line on which $A_1,\ldots,A_n$ are true. That means that $A_1,\ldots,A_n \vdash C$. ■

Because tautological implication is transitive, we can establish tautological implications using several steps. For example, suppose we want to show that $(P \& Q) \vdash (P \vee Q)$. We could do this by using a truth table and showing that $\vdash [(P \& Q) \rightarrow (P \vee Q)]$. But it is much easier to do it as follows. By I1, $(P \& Q) \vdash P$. By I3, $P \vdash (P \vee Q)$. Therefore, by the transitivity of implication, $(P \& Q) \vdash (P \vee Q)$.

Using the transitivity of tautological implication in this way we can string simple implications together to establish more complicated implications. If we want to show that one formula, A, implies another formula, B, we might do this by showing that A implies some other formula and that the second formula implies a third formula (and hence A implies the third formula); then the third formula implies a fourth formula (and hence A implies the fourth formula), and so on until we obtain the formula B. In other words, if we have a string of implications of the form $A \vdash A_1$,

$A_1 \vdash A_2$, $A_2 \vdash A_3, \ldots, A_n \vdash B$, it follows that $A \vdash B$. This is because, as $A \vdash A_1$, and $A_1 \vdash A_2$, we must have $A \vdash A_2$, and then as $A_2 \vdash A_3$, we must have $A \vdash A_3$, and so on until we establish that $A \vdash B$.

Notice that in constructing strings of implications like this, we can also use tautological equivalences, because if two formulas are tautologically equivalent, then each tautologically implies the other. If we want to show that $(P \leftrightarrow Q) \vdash (\sim P \rightarrow \sim Q)$, we may use E7 to observe that

$$(P \leftrightarrow Q) \vdash [(P \rightarrow Q) \ \& \ (Q \rightarrow P)].$$

Then by I2,

$$[(P \rightarrow Q) \ \& \ (Q \rightarrow P)] \vdash (Q \rightarrow P)$$

and by E16,

$$(Q \rightarrow P) \vdash (\sim P \rightarrow \sim Q).$$

Finally, by the transitivity of implication,

$$(P \leftrightarrow Q) \vdash (\sim P \rightarrow \sim Q).$$

Given that tautological implication is transitive, it is easy to see that tautological equivalence is also transitive:

> ***Metatheorem***: If one formula is tautologically equivalent to a second, and the second is tautologically equivalent to a third, then the first is tautologically equivalent to the third.

Proof: Suppose that A eq. B and B eq. C Then $A \vdash B$ and $B \vdash C$, and so by the transitivity of tautological implication, $A \vdash C$. Also $B \vdash A$ and $C \vdash B$, and so once again by the transitivity of tautological implication, $C \vdash A$. Then because both $A \vdash C$ and $C \vdash A$, we can conclude that A eq. C. ■

A further important fact about tautological implication is that it is adjunctive:

> ***Metatheorem***: Given any three formulas, A, B, and C, if $A \vdash B$ and $A \vdash C$, then $A \vdash (B \ \& \ C)$.

Proof: In other words, if a formula implies each of two other formulas, then it implies their conjunction. To see that this is true, suppose $A \vdash B$ and $A \vdash C$. Any assignment of truth values to the atomic parts of A, B, and C that makes A true will make both B and C true, so any such assignment will make $(B \ \& \ C)$ true, and hence $A \vdash (B \ \& \ C)$. ■

To illustrate the use of this metatheorem, suppose we want to show that $P, (P \leftrightarrow Q) \vdash Q$ without constructing a truth table. Clearly

$$P, (P \leftrightarrow Q) \vdash P$$

and

$$P, (P \leftrightarrow Q) \vdash (P \leftrightarrow Q).$$

By E7,

$(P \leftrightarrow Q) \vdash [(P \rightarrow Q) \& (Q \rightarrow P)]$,

and by I1,

$[(P \rightarrow Q) \& (Q \rightarrow P)] \vdash (P \rightarrow Q)$.

Thus by the transitivity of implication,

$P, (P \leftrightarrow Q) \vdash (P \rightarrow Q)$.

Then by the fact that implication is adjunctive,

$P, (P \leftrightarrow Q) \vdash [P \& (P \rightarrow Q)]$.

By I9,

$[P \& (P \rightarrow Q)] \vdash Q$.

Thus by transitivity again,

$P, (P \leftrightarrow Q) \vdash Q$.

We can generalize the adjunctivity of tautological implication somewhat:

> ***Metatheorem***: If a set of formulas $A_1,\ldots,A_n$ tautologically implies each formula in another set $B_1,\ldots,B_m$ then $A_1,\ldots,A_n$ implies the conjunction $(B_1 \& \ldots \& B_m)$ (where the inner parentheses can be in any order).

Proof: Suppose the set $A_1,\ldots,A_n$ implies each of $B_1,\ldots,B_m$. Then as $A_1,\ldots,A_n \vdash B_1$ and $A_1,\ldots,A_n \vdash B_2$, we must have $A_1,\ldots,A_n \vdash (B_1 \& B_2)$. Then as $A_1,\ldots,A_n \vdash B_3$ we must have $A_1,\ldots,A_n \vdash (B_1 \& B_2 \& B_3)$. And so on. Thus $A_1,\ldots,A_n \vdash (B_1 \& \ldots \& B_m)$. ■

We can also generalize the transitivity of tautological implication somewhat:

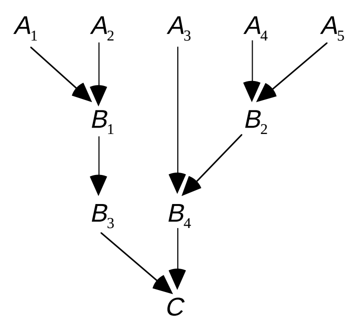
> ***Metatheorem***: If each formula in a set $B_1,\ldots,B_m$ of formulas is tautologically implied by the set of formulas $A_1,\ldots,A_n$. and $B_1,\ldots,B_m \vdash C$, then $A_1,\ldots,A_n \vdash C$.

This is called *strong transitivity*. The form of this principle becomes more obvious if we diagram the relations between the formulas as follows:

$$\left.\begin{array}{c} A_1,\ldots,A_n \vdash B_1 \\ A_1,\ldots,A_n \vdash B_2 \\ \ldots \\ A_1,\ldots,A_n \vdash B_m \end{array}\right\} \vdash C$$

Proof: To establish strong transitivity, suppose each of $B_1,\ldots,B_m$ are implied by $A_1,\ldots,A_n$. Then as we have just seen, $A_1,\ldots,A_n \vdash (B_1 \,\&\, \ldots \,\&\, B_m)$. If $B_1,\ldots,B_m \vdash C$, then $(B_1 \,\&\, \ldots \,\&\, B_m) \vdash C$. So by transitivity, $A_1,\ldots,A_n \vdash C$. The difference between transitivity and strong transitivity is that in the former we have a set of formulas implying a single formula, which in turn implies another formula, whereas in the latter we have a set of formulas implying a second set of formulas, which in turn implies another formula. ■

Strong transitivity is of fundamental importance in understanding the structure of reasoning. Arguments typically have a kind of "tree structure" wherein we start with some premises, draw intermediate conclusions from the premises, draw further intermediate conclusions from the intermediate conclusions, and finally infer our desired conclusion from some of those intermediate steps. For instance, we might have an argument whose structure could be diagrammed as follows:



When we reason like this, we assume that the argument has established that the final conclusion is implied by the initial premises. It is strong transitivity that allows us to conclude that B_3 and B_4 are implied by the initial premises, and then to conclude from that that C is implied by the initial premises. Without strong transitivity, we would have to somehow get the conclusion from the premises in a single step.

Another important fact about implication is the following:

Metatheorem: If $C_1,\ldots,C_n, A \vdash B$, then $C_1,\ldots,C_n \vdash (A \to B)$.

In other words, we can remove one formula from the head of the implication and make it the antecedent of a conditional in the conclusion of the implication. This is *the principle of conditionalization*. It is true for the following reason.

Proof: Suppose $C_1,\ldots,C_n, A \vdash B$. By the definition of tautological implication, this means that the conditional $[(C_1 \,\&\, \ldots \,\&\, C_n \,\&\, A) \to B]$ is a tautology. By E15 this is tautologically equivalent to the conditional

$$[(C_1 \,\&\, \ldots \,\&\, C_n) \to (A \to B)].$$

We have seen that a formula tautologically equivalent to a tautology is itself a tautology, so $\vdash (C_1 \,\&\, \ldots \,\&\, C_n) \to (A \to B)]$. Then $C_1,\ldots,C_n \vdash (A \to B)$. ■

The principle of conditionalization is of considerable use in establishing tautological implications. For example, suppose we want to show that

$$P \vdash [Q \rightarrow (P \,\&\, Q)].$$

By I14,

$$P, Q \vdash (P \,\&\, Q).$$

Then by the principle of conditionalization,

$$P \vdash [Q \rightarrow (P \,\&\, Q)].$$

To take another example, suppose we want to show that

$$(P \rightarrow R) \vdash [(P \vee Q) \rightarrow [(Q \rightarrow R) \rightarrow R]].$$

By I15,

$$(P \vee Q), (P \rightarrow R), (Q \rightarrow R) \vdash R.$$

By the principle of conditionalization,

$$(P \vee Q), (P \rightarrow R) \vdash [(Q \rightarrow R) \rightarrow R].$$

Then by a second application of the principle of conditionalization,

$$(P \rightarrow R) \vdash [(P \vee Q) \rightarrow [(Q \rightarrow R) \rightarrow R]].$$

Combining these facts about tautological implication enables us to give arguments showing that one formula or set of formulas implies another formula. For example, suppose we want to show that

$$(\sim P \vee Q), (\sim Q \vee R) \vdash (\sim P \vee R).$$

We might reason as follows. Clearly

$$(\sim P \vee Q), (\sim Q \vee R) \vdash (\sim P \vee Q)$$

and

$$(\sim P \vee Q), (Q \vee R) \vdash (\sim Q \vee R).$$

By E4,

$$(\sim P \vee Q) \vdash (P \rightarrow Q),$$

and

$$(\sim Q \vee R) \vdash (Q \rightarrow R).$$

So by transitivity,

$$(\sim P \vee Q), (\sim Q \vee R) \vdash (P \rightarrow Q)$$

and

$$(\sim P \vee Q), (\sim Q \vee R) \vdash (Q \rightarrow R).$$

By I13,

$(P \rightarrow Q), (Q \rightarrow R) \vdash (P \rightarrow R)$.

Then by strong transitivity,

$(\sim P \vee Q), (\sim Q \vee R) \vdash (P \rightarrow R)$.

Byy E4,

$(P \rightarrow R) \vdash (\sim P \vee R)$.

Then by transitivity,

$(\sim P \vee Q), (\sim Q \vee R) \vdash (\sim P \vee R)$.

General arguments like this provide an alternative to truth tables in establishing tautologies and tautological implications. However, it can reasonably be doubted whether the construction of such arguments is easier than constructing a truth table. At this point the reader is no doubt wondering how we know which principle to use at any given point in constructing a proof. This question will be answered in the next chapter.

Exercises

Prove the following metatheorem:

1. A truth-functionally inconsistent formula tautologically implies every formula.
2. A tautology is tautologically implied by every formula.
3. Any two tautologies are tautologically equivalent.

9. Failures of Tautologicity

To show that a formula is a tautology, one must either construct an entire truth table or a give a general argument to that effect. This can be difficult. It is generally much easier to show that a formula is not a tautology. It suffices to find a single assignment of truth values (an interpretation) that makes the formula false. Equivalently, one need only find a single line of the truth table on which the formula is false. For example, suppose we want to show that $[(P \rightarrow Q) \,\&\, (Q \rightarrow P)]$ is not a tautology. This is a conjunction, so it suffices to make one conjunct false. For each conjunct, there is only one way to make it false. For example, focusing on the first conjunct, we can make P true and Q false. That makes the whole formula false, showing that it is not a tautology.

A similar technique can be used to show that a proposed tautological implication fails. For example, to show that the two premises $(P \rightarrow \sim Q)$, $(Q \rightarrow \sim R)$ do not jointly imply $(P \rightarrow R)$, it suffices to find an assignment of truth values making the premises true and the conclusion false. The only way to make the conclusion false is to make P true and R false. That automatically makes the first premise true, and if we also make Q false then the second premise is true as well. So the proposed implication does not hold.

Exercises

Find interpretations showing the following:

1. $[P \leftrightarrow (Q \leftrightarrow P)]$ is not a tautology.
2. $[P \leftrightarrow (Q \leftrightarrow P)]$ does not tautologically imply P.
3. $[P \vee (Q \,\&\, \sim R)]$ does not tautologically imply $(P \vee R)$.
